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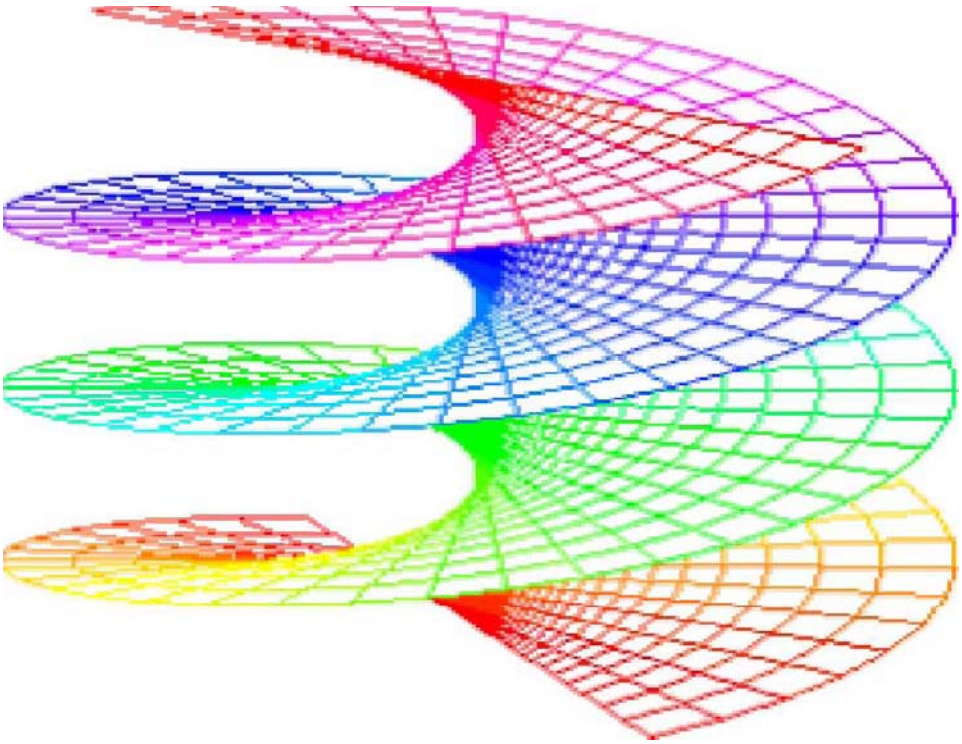

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Contents

A. Majumdar : A note on the Pseudo-Smarandache function 1

S. Gupta : Primes in the Smarandache deconstructive sequence 26

S. Zhang and C. Chen : Recursion formulae for Riemann zeta function and Dirichlet series 31

A. Muktibodh : Smarandache Semiquasi Near-rings 41

J. Sándor : On exponentially harmonic numbers 44

T. Jaíyéọlá : Parastrophic invariance of Smarandache quasigroups 48

M. Karama : Perfect powers in Smarandache n-Expressions 54

T. Jaíyéọlá : Palindromic permutations and generalized Smarandache palindromic permutations 65

J. Sándor : On certain inequalities involving the Smarandache function 78

A. Muktibodh : Sequences of Pentagonal numbers 81

J. Gao : On the additive analogues of the simple function 88

W. Zhu : On the primitive numbers of power p 92

A. Vyawahare : Smarandache sums of products 96

N. Yuan : On the solutions of an equation involving the Smarandache dual function 104

H. Zhou : An infinite series involving the Smarandache power function $SP(n)$ 109

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A note on the Pseudo-Smarandache function

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Abstract This paper gives some results and observations related to the Pseudo-Smarandache function $Z(n)$. Some explicit expressions of $Z(n)$ for some particular cases of n are also given.

Keywords The Pseudo-Smarandache function, Smarandache perfect square, equivalent.

§1. Introduction

The Pseudo-Smarandache function $Z(n)$, introduced by Kashihara [1], is as follows :

Definition 1.1. For any integer $n \geq 1$, $Z(n)$ is the smallest positive integer m such that $1 + 2 + \cdots + m$ is divisible by n. Thus,

$$Z(n) = \min \left\{ m : m \in \mathrm{N} : n \mid \frac{m(m+1)}{2} \right\}. \tag{1.1}$$

As has been pointed out by Ibstedt [2], an equivalent definition of $Z(n)$ is

Definition 1.2.

$$Z(n) = \min\{k : k \in \mathrm{N} : \sqrt{1 + 8kn} \text{ is a perfect square}\}.$$

Kashihara [1] and Ibstedt [2] studied some of the properties satisfied by $Z(n)$. Their findings are summarized in the following lemmas:

Lemma 1.1. For any $m \in N$, $Z(n) \geq 1$. Moreover, $Z(n) = 1$ if and only if $n = 1$, and $Z(n) = 2$ if and only if $n = 3$.

Lemma 1.2. For any prime $p \geq 3$, $Z(p) = p - 1$.

Lemma 1.3. For any prime $p \geq 3$ and any $k \in N$, $Z(p^k) = p^k - 1$.

Lemma 1.4. For any $k \in N$, $Z(2^k) = 2^{k+1} - 1$.

Lemma 1.5. For any composite number $n \geq 4$, $Z(n) \geq \max\{Z(N) : N \mid n\}$.

In this paper, we give some results related to the Pseudo-Smarandache function $Z(n)$.

In §2, we present the main results of this paper. Simple explicit expressions for $Z(n)$ are available for particular cases of n. In Theorems $2.1 - 2.11$, we give the expressions for $Z(2p)$, $Z(3p)$, $Z(2p^2)$, $Z(3p^3)$, $Z(2p^k)$, $Z(3p^k)$, $Z(4p)$, $Z(5p)$, $Z(6p)$, $Z(7p)$ and $Z(11p)$, where p is a prime and $k(\geq 3)$ is an integer. Ibstedt [2] gives an expression for $Z(pq)$ where p and q are distinct primes. We give an alternative expressions for $Z(pq)$, which is more efficient from the computational point of view. This is given in Theorem 2.12, whose proof shows that the solution of $Z(pq)$ involves the solution of two Diophantine equations. Some particular cases of Theorem

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2.12 are given in Corollaries 2.1 – 2.16. We conclude this paper with some observations about the properties of $Z(n)$, given in four Remarks in the last §3.

§2. Main Results

We first state and prove the following results.

Lemma 2.1. Let $n = \frac{k(k+1)}{2}$ for some $k \in N$. Then, $Z(n) = k$.

Proof. Noting that $k(k+1) = m(m+1)$ if and only if $k = m$, the result follows. The following lemma gives lower and upper bounds of $Z(n)$.

Lemma 2.2. $3 \leq n \leq 2n-1$ for all $n \geq 4$.

Proof. Letting $f(m) = \frac{m(m+1)}{2}, m \in N$, see that $f(m)$ is strictly increasing in m with $f(2) = 3$. Thus, $Z(n) = 2$ if and only if $n = 3$. This, together with Lemma 1.1, gives the lower bound of $Z(n)$ for $n \geq 4$. Again, since $n \mid f(2n-1)$, it follows that $Z(n)$ cannot be greater than $2n-1$. Since $Z(n) = 2n-1$ if $n = 2k$ for some $k \in N$, it follows that the upper bound of $Z(n)$ in Lemma 2.2 cannot be improved further. However, the lower bound of $Z(n)$ can be improved. For example, since $f(4) = 10$, it follows that $Z(n) \geq 5$ for all $n \geq 11$. A better lower bound of $Z(n)$ is given in Lemma 1.5 for the case when n is a composite number. In Theorems 2.1 – 2.4, we give expressions for $Z(2p), Z(3p), Z(2p^2)$ and $Z(3p^2)$ where $p \geq 5$ is a prime. To prove the theorems, we need the following results.

Lemma 2.3. Let p be a prime. Let an integer $n(\geq p)$ be divisible by p^k for some integer $k(\geq 1)$. Then, p^k does not divide $n+1$ (and $n-1$).

Lemma 2.4. $6 \mid n(n+1)(n+2)$ for any $n \in N$. In particular, $6 \mid (p^2-1)$ for any prime $p \geq 5$.

Proof. The first part is a well-known result. In particular, for any prime $p \geq 5$, $6 \mid (p-1)p(p+1)$. But since $p(\geq 5)$ is not divisible by 6, it follows that $6 \mid (p-1)(p+1)$.

Theorem 2.1. If $p \geq 5$ is a prime, then

$$Z(2p) = \begin{cases} p-1, & \text{if } 4 \mid (p-1); \\ p, & \text{if } 4 \mid (p+1). \end{cases}$$

Proof.

$$Z(2p) = \min\left\{m : 2p \mid \frac{m(m+1)}{2}\right\} = \min\left\{m : p \mid \frac{m(m+1)}{4}\right\}. \tag{1}$$

If $p \mid m(m+1)$, then p must divide either m or $m+1$, but not both (by Lemma 2.3). Thus, the minimum m in (1) may be taken as $p-1$ or p depending on whether $p-1$ or $p+1$ respectively is divisible by 4. We now consider the following two cases that may arise :

Case 1 : p is of the form p=4a+1 for some integer $a \geq 1$. In this case, $4 \mid (p-1)$, and hence, $Z(2p) = p-1$.

Case 2 : p is of the form $p = 4a+3$ for some integer $a \geq 1$. Here, $4 \mid (p+1)$ and hence, $Z(2p) = p$.

Theorem 2.2. If $p \geq 5$ is a prime, then

$$Z(3p) = \begin{cases} p-1, & \text{if } 3 \mid (p-1); \\ p, & \text{if } 3 \mid (p+1). \end{cases}$$

Proof.

$$Z(3p) = \min\left\{m : 3p \mid \frac{m(m+1)}{2}\right\} = \min\left\{m : p \mid \frac{m(m+1)}{6}\right\}. \tag{2}$$

If $p \mid m(m+1)$, then p must divide either m or m+1, but not both (by Lemma 2.3). Thus, the minimum m in (2) may be taken as $p-1$ or p according as $p-1$ or $p+1$ respectively is divisible by 6. But, since both $p-1$ and $p+1$ are divisible by 2, it follows that the minimum m in (2) may be taken as $p-1$ or p according as $p-1$ or $p+1$ respectively is divisible by 3.

We now consider the following two possible cases that may arise :

Case 1 : p is of the form $p = 3a+1$ for some integer $a \geq 1$. In this case, $3 \mid (p-1)$, and hence, $Z(3p) = p-1$.

Case 2 : p is of the form $p = 3a+2$ for some integer $a \geq 1$. Here, $3 \mid (p+1)$, and hence, $Z(3p) = p$.

Theorem 2.3. If $p \geq 3$ is a prime, then $Z(2p^2) = p^2 - 1$.

Proof.

$$Z(2p^2) = \min\left\{m : 2p^2 \mid \frac{m(m+1)}{2}\right\} = \min\left\{m : p^2 \mid \frac{m(m+1)}{4}\right\}. \tag{3}$$

If $p^2 | m(m+1)$, then p^2 must divide either m or $m+1$, but not both (by Lemma 2.3). Thus, the minimum m in (3) may be taken as p^2-1 if p^2-1 is divisible by 4. But, since both $p-1$ and $p+1$ are divisible by 2, it follows that $4 \mid (p-1)(p+1)$. Hence, $Z(2p^2) = p^2 - 1$.

Theorem 2.4. If $p \geq 5$ is a prime, then $Z(3p^2) = p^2 - 1$.

Proof.

$$Z(3p^2) = \min\left\{m : 3p^2 \mid \frac{m(m+1)}{2}\right\} = \min\left\{m : p^2 \mid \frac{m(m+1)}{6}\right\}. \tag{4}$$

If $p^2 | m(m+1)$, then p^2 must divide either m or $m+1$, but not both (by Lemma 2.3). Thus, the minimum m in (4) may be taken as $p^2 1$ if p^2-1 is divisible by 6. By Lemma 2.4, $6 \mid (p^2-1)$. Consequently, $Z(2p^2) = p^2 - 1$.

Definition 2.1. A function $g : \mathrm{N} \to \mathrm{N}$ is called multiplicative if and only if $g(n_1 n_2) = g(n_1)g(n_2)$ for all $n_1, n_2 \in N$ with $(n_1, n_2) = 1$.

Remark 2.1. From Lemma 1.2 and Theorem 2.1, we see that $Z(2p) \neq 3(p-1) = Z(2)Z(p)$ for any odd prime p. Moreover, $Z(3p^2) = p^2 - 1 \neq Z(2p^2) + Z(p^2)$. These show that $Z(n)$ is neither additive nor multiplicative, as has already been noted by Kashihara [1]. The expressions for $Z(2p^k)$ and $Z(3p^k)$ for $k \geq 3$ are given in Theorem 2.5 and Theorem 2.6 respectively. For the proofs, we need the following results:

Lemma 2.5.

(1) 4 divides $3^2 k - 1$ for any integer $k \geq 1$.

(2) 4 divides $3^{2k+1} + 1$ for any integer $k \geq 0$.

Proof.

(1) Writing $3^{2k} - 1 = (3k - 1)(3k + 1)$, the result follows immediately.

(2) The proof is by induction on k. The result is clearly true for $k = 0$. So, we assume that the result is true for some integer k, so that 4 divides $3^{2k+1} + 1$ for some k. Now, since $3^{2k+3} + 1 = 9(3^{2k+1} + 1) - 8$, it follows that 4 divides $3^{2k+3} + 1$, completing the induction.

Lemma 2.6.

(1) 3 divides $2^2k - 1$ for any integer $k \geq 1$.

(2) 3 divides $2^{2k+1} + 1$ for any integer $k \geq 0$.

Proof.

(1) By Lemma 2.4, 3 divides $(2k-1)2k(2k+1)$. Since 3 does not divide $2k$, 3 must divide $(2k - 1)(2k + 1) = 2^2k - 1$.

(2) The result is clearly true for $k = 0$. To prove by induction, the induction hypothesis is that 3 divides $2^{2k+1} + 1$ for some k. Now, since $2^{2k+3} + 1 = 4(3^{2k+1} + 1) - 3$, it follows that 3 divides $2^{2k+3} + 1$, so that the result is true for $k + 1$ as well, completing the induction.

Theorem 2.5. If $p \geq 3$ is a prime and $k \geq 3$ is an integer, then

$$Z(2p^k) = \begin{cases} p^k, & \text{if } 4 \mid (p-1) \text{ and } k \text{ is odd;} \\ p^k - 1, & \text{otherwise.} \end{cases}$$

Proof.

$$Z(2p^k) = \min\left\{m : 2p^k \mid \frac{m(m+1)}{2}\right\} = \min\left\{m : p^k \mid \frac{m(m+1)}{4}\right\}. \tag{5}$$

If $p^k | m(m+1)$, then p^k must divide either m or $m+1$, but not both (by Lemma 2.3). Thus, the minimum m in (5) may be taken as $p^k - 1$ or p^k according as $p^k - 1$ or p^k is respectively divisible by 4. We now consider the following two possibilities:

Case 1 : p is of the form $4a + 1$ for some integer $a \geq 1$. In this case, $p^k = (4a + 1)^k = (4a)^k + C_k^1(4a)^{k-1} + \cdots + C_k^{k-1}(4a) + 1$, showing that $4 \mid (p^k - 1)$. Hence, in this case, $Z(2p^k) = p^k - 1$.

Case 2 : p is of the form $4a + 3$ for some integer $a \geq 1$. Here, $p^k = (4a + 3)k = (4a)^k + C_k^1(4a)^{k-1}3 + \cdots + C_k^{k-1}(4a)3^{k-1} + 3^k$.

(1) If $k \geq 2$ is even, then by Lemma 2.5, $4 \mid (3^k - 1)$, so that $4 \mid (p^k - 1)$. Thus, $Z(2p^k) = p^k - 1$.

(2) If $k \geq 3$ is odd, then by Lemma 2.5, $4 \mid (3^k+1)$, and so $4 \mid (p^k+1)$. Hence, $Z(2p^k) = p^k$. All these complete the proof of the theorem.

Theorem 2.6. If $p \geq 3$ is a prime and $k \geq 3$ is an integer, then

$$Z(3p^k) = \begin{cases} p^k, & \text{if } 3 \mid (p+1) \text{ and } k \text{ is odd;} \\ p^k - 1, & \text{otherwise.} \end{cases}$$

Proof.

$$Z(3p^k) = \min\left\{m : 3p^k \mid \frac{m(m+1)}{2}\right\} = \min\left\{m : p^k \mid \frac{m(m+1)}{6}\right\}. \tag{6}$$

If $p^k|m(m+1)$, then p^k must divide either m or $m+1$, but not both (by Lemma 2.3). Thus, the minimum m in (6) may be taken as p^k-1 or p^k according as p^k-1 or p^k is respectively divisible by 6. We now consider the following two possible cases:

Case 1 : p is of the form $3a+1$ for some integer $a \geq 1$. In this case, $p^k = (3a+1)^k = (3a)^k + C_k^1(3a)^{k-1} + \cdots + C_k^{k-1}(3a) + 1$, it follows that $3 \mid (p^k-1)$. Thus, in this case, $Z(3p^k) = p^k - 1$.

Case 2 : p is of the form $3a+2$ for some integer $a \geq 1$. Here, $p^k = (3a+2)k = (3a)^k + C_k^1(3a)^{k-1}(2) + \cdots + C_k^{k-1}(3a)2^{k-1} + 2^k$.

(1) If $k \geq 2$ is even, then by Lemma 2.6, $3 \mid (2^k-1)$, so that $3 \mid (p^k-1)$. Thus, $Z(3p^k) = p^k - 1$.

(2) If $k \geq 3$ is odd, then by Lemma 2.6, $3 \mid (2^k+1)$, and so $3 \mid (p^k+1)$. Thus, $Z(3p^k) = p^k$.

In Theorem 2.7 - Theorem 2.9, we give the expressions for $Z(4p)$, $Z(5p)$ and $Z(6p)$ respectively, where p is a prime. Note that, each case involves 4 possibilities.

Theorem 2.7. If $p \geq 5$ is a prime, then

$$Z(4p) = \begin{cases} p-1, & \text{if } 8 \mid (p-1); \\ p, & \text{if } 8 \mid (p+1); \\ 3p-1, & \text{if } 8 \mid (3p-1); \\ 3p, & \text{if } 8 \mid (3p+1). \end{cases}$$

Proof.

$$Z(4p) = \min\left\{m : 4p \mid \frac{m(m+1)}{2}\right\} = \min\left\{m : p \mid \frac{m(m+1)}{8}\right\}. \tag{7}$$

If $p|m(m+1)$, then p must divide either m or $m+1$, but not both (by Lemma 2.3), and then 8 must divide either $p-1$ or $p+1$, In the particular case when 8 divides $p-1$ or $p+1$, the minimum m in (7) may be taken as $p-1$ or $p+1$ respectively. We now consider the following four cases may arise:

Case 1 : p is of the form $p = 8a+1$ for some integer $a \geq 1$. In this case, $8 \mid (p-1)$, and hence $Z(4p) = p-1$.

Case 2 : p is of the form $p = 8a+7$ for some integer $a \geq 1$. Here, $8 \mid (p+1)$, and hence $Z(4p) = p$.

Case 3 : p is of the form $p = 8a+3$ for some integer $a \geq 1$. In this case, $8 \mid (3p-1)$, and hence $Z(4p) = 3p-1$.

Case 4 : p is of the form $p = 8a+5$ for some integer $a \geq 1$. Here, $8 \mid (3p+1)$, and hence $Z(4p) = 3p$.

Theorem 2.8. If $p \geq 7$ is a prime, then

$$Z(5p) = \begin{cases} p-1, & \text{if } 10 \mid (p-1); \\ p, & \text{if } 10 \mid (p+1); \\ 2p-1, & \text{if } 5 \mid (2p-1); \\ 2p, & \text{if } 5 \mid (2p+1). \end{cases}$$

Proof.

$$Z(5p) = \min\left\{m : 5p \mid \frac{m(m+1)}{2}\right\} = \min\left\{m : p \mid \frac{m(m+1)}{10}\right\}. \tag{8}$$

If $p|m(m+1)$, then p must divide either m or $m+1$, but not both (by Lemma 2.3), and then 5 must divide either $m-1$ or $m+1$, In the particular case when 5 divides $p-1$ or $p+1$, the minimum m in (8) may be taken as $p-1$ or $p+1$ respectively. We now consider the four below that may arise:

Case 1 : p is a prime whose last digit is 1. In this case, $10 \mid (p-1)$, and hence $Z(5p) = p-1$.

Case 2 : p is a prime whose last digit is 9. In such a case, $10 \mid (p+1)$, and so $Z(5p) = p$.

Case 3 : p is a prime whose last digit is 3. In this case, $5 \mid (2p-1)$. Thus, the minimum m in (9) may be taken as $2p-1$. Hence $Z(5p) = 2p-1$.

Case 4 : p is a prime whose last digit is 7. Here, $5 \mid (2p+1)$, and hence $Z(5p) = 2p$.

Theorem 2.9. If $p \geq 5$ is a prime, then

$$Z(6p) = \begin{cases} p-1, & \text{if } 12 \mid (p-1); \\ p, & \text{if } 12 \mid (p+1); \\ 2p-1, & \text{if } 4 \mid (3p+1); \\ 2p, & \text{if } 4 \mid (3p-1). \end{cases}$$

Proof.

$$Z(6p) = \min\left\{m : 6p \mid \frac{m(m+1)}{2}\right\} = \min\left\{m : p \mid \frac{m(m+1)}{12}\right\}. \tag{9}$$

If $p|m(m+1)$, then p must divide either m or $m+1$, but not both (by Lemma 2.3), and then 12 must divide either $m-1$ or $m+1$, In the particular case when 12 divides $p-1$ or $p+1$, the minimum m in (9) may be taken as $p-1$ or p respectively. We now consider the four cases that may arise:

Case 1 : p is of the form $p = 12a+1$ for some integer $a \geq 1$. In this case, $12 \mid (p-1)$, and hence $Z(6p) = p-1$.

Case 2 : p is of the form $p = 12a+11$ for some integer $a \geq 1$. Here, $12 \mid (p+1)$, and hence $Z(6p) = p$.

Case 3 : p is of the form $p = 12a+5$ for some integer $a \geq 1$. In this case, $4 \mid (3p+1)$. The minimum m in (10) may be taken as $3p$, and hence $Z(6p) = 3p$.

Case 4 : p is of the form $p = 12a+7$ for some integer $a \geq 1$. Here, $4 \mid (3p-1)$, and hence $Z(6p) = 3p-1$.

It is possible to find explicit expressions for $Z(7p)$ or $Z(11p)$, where p is a prime, as are given in Theorem 2.10 and Theorem 2.11 respectively, but it becomes more complicated. For example, in finding the expression for $Z(7p)$, we have to consider all the six possibilities, while the expression for $Z(11p)$ involves 10 alternatives.

Theorem 2.10. If $p \geq 11$ is a prime, then

$$Z(7p) = \begin{cases} p-1, & \text{if } 7 \mid (p-1); \\ p, & \text{if } 7 \mid (p+1); \\ 2p-1, & \text{if } 7 \mid (2p-1); \\ 2p, & \text{if } 7 \mid (2p+1); \\ 3p-1, & \text{if } 7 \mid (3p-1); \\ 3p, & \text{if } 7 \mid (3p+1). \end{cases}$$

Proof:

$$Z(7p) = \min\{m : 7p | \frac{m(m+1)}{2}\} = \min\{m : p | \frac{m(m+1)}{14}\}. \tag{10}$$

If $p|m(m+1)$, then p must divide either m or $m+1$, but not both (by Lemma 2.3), and then 7 must divide either $m+1$ or m respectively. In the particular case when 12 divides $p-1$ or $p+1$, the minimum m in (10) may be taken as $p-1$ or p respectively. We now consider the following six cases that may arise:

Case 1 : p is of the form $p = 7a+1$ for some integer $a \geq 1$. In this case, $7|(p-1)$. Therefore, $Z(7p) = p-1$.

Case 2 : p is of the form $p = 7a+6$ for some integer $a \geq 1$. Here, $7|(p+1)$, and so, $Z(7p) = p$.

Case 3 : p is of the form $p = 7a+2$ for some integer $a \geq 1$, so that $7|(3p+1)$. In this case, the minimum m in (11) may be taken as $3p$. That is, $Z(7p) = 3p$.

Case 4 : p is of the form $p = 7a+5$ for some integer $a \geq 1$. Here, $7|(3p-1)$, and hence, $Z(7p) = 3p-1$.

Case 5 : p is of the form $p = 7a+3$ for some integer $a \geq 1$. In this case, $7|(2p+1)$, and hence, $Z(7p) = 2p$.

Case 6 : p is of the form $p = 7a+4$ for some integer $a \geq 1$. Here, $7|(2p-1)$, and hence, $Z(7p) = 2p-1$.

Theorem 2.11. For any prime $p \geq 13$,

$$Z(7p) = \begin{cases} p-1, & \text{if } 11 \mid (p-1); \\ p, & \text{if } 11 \mid (p+1); \\ 2p-1, & \text{if } 11 \mid (2p-1); \\ 2p, & \text{if } 11 \mid (2p+1); \\ 3p-1, & \text{if } 11 \mid (3p-1); \\ 3p, & \text{if } 11 \mid (3p+1); \\ 4p-1, & \text{if } 11 \mid (4p-1); \\ 4p, & \text{if } 11 \mid (4p+1); \\ 5p-1, & \text{if } 11 \mid (5p-1); \\ 5p, & \text{if } 11 \mid (5p+1). \end{cases}$$

,

Proof:

$$Z(11p) = \min\{m : 11p|\frac{m(m+1)}{2}\} = \min\{m : p|\frac{m(m+1)}{22}\}. \quad (11)$$

If $p|m(m+1)$, then p must divide either m or $m+1$, but not both (by Lemma 2.3), and then 11 must divide either $m+1$ or m respectively. In the particular case when 11 divides $p-1$ or $p+1$, the minimum m in (11) may be taken as $p-1$ or p respectively. We have to consider the ten possible cases that may arise :

Case 1 : p is of the form $p = 11a+1$ for some integer $a \geq 1$. In this case, $11|(p-1)$, and so, $Z(11p) = p-1$.

Case 2 : p is of the form $p = 11a+10$ for some integer $a \geq 1$. Here, $11|(p+1)$, and hence, $Z(11p) = p$.

Case 3 : p is of the form $p = 11a+2$ for some integer $a \geq 1$. In this case, $11|(5p+1)$, and hence, $Z(11p) = 5p$.

Case 4 : p is of the form $p = 11a+9$ for some integer $a \geq 1$. Here, $11|(5p-1)$, and hence, $Z(11p) = 5p-1$.

Case 5 : p is of the form $p = 11a+3$ for some integer $a \geq 1$. In this case, $11|(4p-1)$, and hence, $Z(11p) = 4p-1$.

Case 6 : p is of the form $p = 11a+8$ for some integer $a \geq 1$. Here, $11|(4p+1)$, and hence, $Z(11p) = 4p$.

Case 7 : p is of the form $p = 11a+4$ for some integer $a \geq 1$. In this case, $11|(3p-1)$, and hence, $Z(11p) = 3p-1$.

Case 8 : p is of the form $p = 11a+7$ for some integer $a \geq 1$. Here, $11|(3p+1)$, and hence, $Z(11p) = 3p$.

Case 9 : p is of the form $p = 11a+5$ for some integer $a \geq 1$. In this case, $11|(2p+1)$, and hence, $Z(11p) = 2p$.

Case 10 : p is of the form $p = 11a+6$ for some integer $a \geq 1$. Here, $11|(2p-1)$, and hence, $Z(11p) = 2p-1$.

In Theorem 2.12, we give an expression for $Z(pq)$, where p and q are two distinct primes. In this connection, we state the following lemma. The proof of the lemma is similar to, for example, Theorem 12.2 of Gioia [3], and is omitted here.

Lemma 2.7. Let p and q be two distinct primes. Then, the Diophantine equation

$$qy - px = 1$$

has an infinite number of solutions. Moreover, if (x_0, y_0) is a solution of the Diophantine equation, then any solution is of the form

$$x = x_0 + qt, y = y_0 + pt,$$

where $t \geq 0$ is an integer.

Theorem 2.12. Let p and q be two primes with $q > p \geq 5$. Then,

$$Z(pq) = \min\{qy_0 - 1, px_0 - 1\},$$

where

$$y_0 = \min\{y : x, y \in N, qy - px = 1\},$$

$$x_0 = \min\{x : x, y \in N, px - qy = 1\}.$$

Proof: Since

$$Z(pq) = \min\{m : pq|\frac{m(m+1)}{2}\}, \tag{12}$$

it follows that we have to consider the three cases below that may arise :

Case 1 : When $p|m$ and $q|(m+1)$. In this case, $m = px$ for some integer $x \geq 1$, $m+1 = qy$ for some integer $y \geq 1$. From these two equations, we get the Diophantine equation

$$qy - px = 1.$$

By Lemma 2.7, the above Diophantine equation has infinite number of solutions. Let

$$y_0 = \min\{y : x, y \in N, qy - px = 1\}.$$

For this y_0, the corresponding x_0 is given by the equation $q_0y - p_0x = 1$. Note that y_0 and x_0 cannot be both odd or both even. Then, the minimum m in (12) is given by

$$m + 1 = qy_0 \Rightarrow m = qy_0 - 1.$$

Case 2 : When $p|(m+1)$ and $q|m$. Here, $m + 1 = px$ for some integer $x \geq 1$, $m = qy$ for some integer $y \geq 1$. These two equations lead to the Diophantine equation. $px - qy = 1$. Let

$$x_0 = \min\{x : x, y \in N, px - qy = 1\}.$$

For this x_0, the corresponding y_0 is given by $y_0 = (px_0 - 1)/q$. Here also, x_0 and y_0 both cannot be odd or even simultaneously. The minimum m in (12) is given by

$$m + 1 = px_0 \Rightarrow m = px_0 - 1.$$

Case 3 : When $pq|(m+1)$. In this case, $m = pq-1$. But then, by Case 1 and Case 2 above, this does not give the minimum m. Thus, this case cannot occur. The proof of the theorem now follows by virtue of Case 1 and Case 2.

Remark 2.2. Let p and q be two primes with $q \geq p \geq 5$. Let $q = kp + \ell$ for some integers k and ℓ with $k \geq 1$ and $1 \leq \ell \leq p - 1$. We now consider the two cases given in Theorem 2.12 :

Case 1 : When $p|m$ and $q|(m + 1)$. In this case, $m = px$ for some integer $x \geq 1$, $m + 1 = qy = (kp + \ell)y$ for some integer $y \geq 1$. From these two equations, we get

$$\ell y - (x - ky)p = 1 \tag{2.1}.$$

Case 2 : When $p|(m+1)$ and $q|m$. Here, $m+1 = px$ for some integer $x \geq 1$, $m = (kp+\ell)y$ for some integer $y \geq 1$. These two equations lead to

$$(x - ky)p - \ell y = 1 \tag{2.2}.$$

In some particular cases, explicit expressions of $Z(pq)$ may be found. These are given in the following corollaries.

Corollary 2.1. Let p and q be two primes with $q > p \geq 5$. Let $q = kp+1$ for some integer $k \geq 2$. Then, $Z(pq) = q - 1$.

Proof. From (2.1) with $\ell = 1$, we get $y - (x - ky)p = 1$, the minimum solution of which is $y = 1$, $x = ky = k$. Then, the minimum m in (12) is given by

$$m + 1 = qy = q \Rightarrow m = q - 1.$$

Note that, from (2.2) with $\ell = 1$, we have $(x - ky)p - y = 1$, with the least possible solution $y = p - 1$ (and $x - ky = 1$).

Corollary 2.2. Let p and q be two primes with $q > p \geq 5$. Let $q = (k+1)p - 1$ for some integer $k \geq 1$.

Then, $Z(pq) = q$.

Proof. From (2.2) with $\ell = p - 1$, we have, $y - [(k+1)y - x]p = 1$, the minimum solution of which is $y = 1$, $x = (k+1)y = k+1$. Then, the minimum m in (12) is given by $m = qy = q$. Note that, from (2.1) with $\ell = p - 1$, we have $[(k+1)y - x]p - x = 1$ with the least possible solution $y = p - 1$ (and $(k+1)y - x = 1$).

Corollary 2.3. Let p and q be two primes with $q > p \geq 5$. Let $q = kp+2$ for some integer $k \geq 1$. Then,

$$Z(pq) = \frac{q(p-1)}{2}.$$

Proof. From (2.2) with $\ell = 2$, we have $(x - ky)p - 2y = 1$, with the minimum solution $y = \frac{p-1}{2}$ (and $x - ky = 1$). This gives $m = qy = \frac{q(p-1)}{2}$ as one possible solution of (12). Now, (2.1) with $\ell = 2$ gives $2y - (x - ky)p = 1$, with the minimum solution $y = \frac{p+1}{2}$ (and $x = ky + 1$). This gives $m = qy - 1 = \frac{q(p+1)}{2} - 1$ as another possible solution of (12). Now, since $\frac{q(p+1)}{2} - 1 > \frac{q(p-1)}{2}$, it follows that

$$Z(pq) = \frac{q(p-1)}{2},$$

which we intended to prove.

Corollary 2.4. Let p and q be two primes with $q > p \geq 5$. Let $q = (k+1)p - 2$ for some integer $k \geq 1$. Then,

$$Z(pq) = \frac{q(p-1)}{2} - 1.$$

Proof. By (2.1) with $\ell = p - 2$, we get $[(k+1)y - x]p - 2y = 1$, whose minimum solution is $y = \frac{p-1}{2}$(and $x = (k+1)y - 1$). This gives $m = qy - 1 = \frac{q(p-1)}{2} - 1$ as one possible solution of (12). Note that, (2.2) with $\ell = p - 2$ gives $2y - [(k+1)y - x]p = 1$, with the minimum solution $y = \frac{p+1}{2}$ (and $x = (k+1)y - 1$). Corresponding to this case, we get $m = qy = \frac{q(p+1)}{2}$ as another possible solution of (12). But since $\frac{q(p+1)}{2} > \frac{q(p-1)}{2} - 1$, it follows that $Z(pq) = \frac{q(p-1)}{2} - 1$, establishing the theorem.

Corollary 2.5. Let p and q be two primes with $q > p \geq 7$. Let $q = kp+3$ for some integer $k \geq 1$. Then,

$$Z(pq) = \begin{cases} \frac{q(p-1)}{3}, & \text{if } 3|(p-1); \\ \frac{q(p+1)}{3} - 1, & \text{if } 3|(p+1). \end{cases}$$

Proof. From (2.1) and (2.2) with $\ell = 3$, we have respectively

$$3y - (x - ky)p = 1, \tag{13}$$

$$(x - ky)p - 3y = 1. \tag{14}$$

We now consider the following two possible cases :

Case 1 : When 3 divides $p - 1$.

In this case, the minimum solution is obtained from (14), which is $y = \frac{p-1}{3}$(and $x-ky = 1$). Also, $p - 1$ is divisible by 2 as well. Therefore, the minimum m in (12) is $m = qy = \frac{q(p-1)}{3}$.

Case 2 : When 3 divides $p + 1$.

In this case, (13) gives the minimum solution, which is $y = \frac{p+1}{3}$(and $x - ky = 1$). Note that, 2 divides $p + 1$. Therefore, the minimum m in (12) $m = qy - 1 = \frac{q(p+1)}{3} - 1$.

Thus, the theorem is established.

Corollary 2.6. Let p and q be two primes with $q > p \geq 7$. Let $q = (k + 1)p - 3$ for some integer $k \geq 1$. Then,

$$Z(pq) = \begin{cases} \frac{q(p+1)}{3}, & \text{if } 3|(p+1); \\ \frac{q(p-1)}{3} - 1, & \text{if } 3|(p-1). \end{cases}$$

Proof. From (2.1) and (2.2) with $\ell = p - 3$, we have respectively

$$[(k + 1)y - x]p - 3y = 1, \tag{15}$$

$$3y - [(k + 1)y - x]p = 1. \tag{16}$$

We now consider the following two cases :

Case 1 : When 3 divides $p + 1$.

In this case, the minimum solution, obtained from(14), is $y = \frac{p+1}{3}$(and $x = (k + 1)y - 1$).Moreover, 2 divides $p + 1$. Therefore, the minimum m in (12) is $m = qy = \frac{q(p+1)}{3}$.

Case 2 : When 3 divides $p - 1$.

In this case, the minimum solution, obtained from (13), is $y = \frac{p-1}{3}$(and $x = (k + 1)y - 1$). Moreover, 2 divides $p - 1$. Therefore, the minimum m in (12) is $m = qy = \frac{q(p-1)}{3} - 1$.

Corollary 2.7. Let p and q be two primes with $q > p \geq 7$. Let $q = kp+4$ for some integer $k \geq 1$. Then,

$$Z(pq) = \begin{cases} \frac{q(p-1)}{4}, & \text{if } 4|(p-1); \\ \frac{q(p+1)}{4} - 1, & \text{if } 4|(p+1). \end{cases}$$

Proof. From (2.1) and (2.2) with $\ell = 4$, we have respectively

$$4y - (x - ky)p = 1, \tag{17}$$

$$(x - ky)p - 4y = 1. \tag{18}$$

Now, for any prime $p \geq 7$, exactly one of the following two cases can occur : Either $p - 1$ is divisible by 4, or $p+1$ is divisible by 4. We thus consider the two possibilities separately below:

Case 1 : When 4 divides $p - 1$.

In this case, the minimum solution is obtained from (18), is $y = \frac{p-1}{4}$(and $x = ky + 1$). Therefore, the minimum m in (12) is is $m = qy = \frac{q(p-1)}{4}$.

Case 2 : When 4 divides $p + 1$.

In this case, (17) gives the minimum solution, which is $y = \frac{p+1}{4}$(and $x = ky+1$). Therefore, the minimum m in (12) $m = qy - 1 = \frac{q(p+1)}{4} - 1$.

Corollary 2.8. Let p and q be two primes with $q > p \geq 7$. Let $q = (k + 1)p - 4$ for some integer $k \geq 1$. Then,

$$Z(pq) = \begin{cases} \frac{q(p+1)}{4}, & \text{if } 4|(p+1); \\ \frac{q(p-1)}{4} - 1, & \text{if } 4|(p-1). \end{cases}$$

Proof. From (2.1) and (2.2) with $\ell = p - 4$, we have respectively

$$[(k + 1)y - x]p - 4y = 1, \tag{19}$$

$$4y - [(k + 1)y - x]p = 1. \tag{20}$$

We now consider the following two cases which are the only possibilities (as noted in the proof of Corollary 2.7).

Case 1 : When 4 divides $p + 1$.

In this case, the minimum solution obtained from (20) is $y = \frac{p+1}{4}$ (and $x = (k + 1)y - 1$).Therefore, the minimum m in (12) is $m = qy = \frac{q(p+1)}{4}$.

Case 2 : When 4 divides $p - 1$.

In this case, the minimum solution, obtained from (19),is $y = \frac{p-1}{4}$ (and $x = (k + 1)y - 1$).Therefore, the minimum m in (12) is $m = qy = \frac{q(p-1)}{4} - 1$.

Corollary 2.9 . Let p and q be two primes with $q > p \geq 11$. Let $q = kp + 5$ for some integer $k \geq 1$. Then,

$$Z(pq) = \begin{cases} \frac{q(p-1)}{5}, & \text{if } 5|(p-1); \\ q(2a + 1) - 1, & \text{if } p = 5a + 2; \\ q(2a + 1), & \text{if } p = 5a + 3; \\ \frac{q(p+1)}{5} - 1, & \text{if } 5|(p+1). \end{cases}$$

Proof. From (2.1) and (2.2) with $\ell = 5$, we have respectively

$$5y - (x - ky)p = 1, \tag{21}$$

$$(x - ky)p - 5y = 1. \tag{22}$$

Now, for any prime $p \geq 7$, exactly one of the following four cases occur:

Case 1 : When p is of the form $p = 5a + 1$ for some integer $a \geq 2$.

In this case, 5 divides $p - 1$. Then, the minimum solution is obtained from (22) which is $y = \frac{p-1}{5}$ (and $x - ky = 1$). Therefore, the minimum m in (12) is $m = qy = \frac{q(p-1)}{5}$.

Case 2 : When p is of the form $p = 5a + 2$ for some integer $a \geq 2$.

In this case, from (21) and (22), we get respectively

$$1 = 5y - (x - ky)(5a + 2) = 5[y - (x - ky)a] - 2(x - ky), \tag{23}$$

$$1 = (x - ky)(5a + 2) - 5y = 2(x - ky) - 5[y - (x - ky)a] \tag{24}$$

Clearly, the minimum solution is obtained from (23), which is

$y - (x - ky)a = 1, \quad x - ky = 2 \Longrightarrow y = 2a + 1$ (and $x = k(2a + 1) + 2$).

Hence, in this case, the minimum m in (12) is $m = qy - 1 = q(2a + 1) - 1$.

Case 3 : When p is of the form $p = 5a + 3$ for some integer $a \geq 2$. From (21) and (22), we get

$$1 = 5y - (x - ky)(5a + 3) = 5[y - (x - ky)a] - 3(x - ky), \tag{25}$$

$$1 = (x - ky)(5a + 3) - 5y = 3(x - ky) - 5[y - (x - ky)a]. \tag{26}$$

The minimum solution is obtained from (27) as follows :

$y - (x - ky)a = 1, \quad x - ky = 2 \Longrightarrow y = 2a + 1$ (and $x = k(2a + 1) + 2$).

Hence, in this case, the minimum m in (12) is $m = qy = q(2a + 1)$.

Case 4 : When p is of the form $p = 5a + 4$ for some integer $a \geq 2$.

In this case, 5 divides $p + 1$. Then, the minimum solution is obtained from (21), which is $y = \frac{p+1}{5}$ (and $x - ky = 1$). Therefore, the minimum m in (12) is $m = qy - 1 = \frac{q(p+1)}{5} - 1$.

Corollary 2.10. Let p and q be two primes with $q > p \geq 11$. Let $q = (k + 1)p - 5$ for some integer $k \geq 1$. Then,

$$Z(pq) = \begin{cases} \frac{q(p-1)}{5} - 1, & \text{if } 5|(p-1); \\ q(2a+1), & \text{if } p = 5a+2; \\ q(2a+1) - 1, & \text{if } p = 5a+3; \\ \frac{q(p+1)}{5}, & \text{if } 5|(p+1). \end{cases}$$

Proof. From (2.1) and (2.2) with $\ell = p - 5$, we have respectively

$$[(k + 1)y - x]p - 5y = 1, \tag{27}$$

$$5y - [(k + 1)y - x]p = 1. \tag{28}$$

As in the proof of Corollary 2.9, we consider the following four possibilities :

Case 1 : When p is of the form $p = 5a + 1$ for some integer $a \geq 2$.

In this case, 5 divides $p - 1$. Then, the minimum solution is obtained from (27), which is $y = \frac{p-1}{5}$ (and $x = (k+1)y-1$). Therefore, the minimum m in (12) is $m = qy-1 = \frac{q(p-1)}{5}-1$.

Case 2 : When p is of the form $p = 5a + 2$ for some integer $a \geq 2$.

In this case, from (27) and (28), we get respectively

$$1 = [(k + 1)y - x](5a + 2) - 5y = 2[(k + 1)y - x] - 5[y - a(k + 1)y - x], \tag{29}$$

$$1 = 5y - [(k + 1)y - x](5a + 2) = 5[y - a(k + 1)y - x] - 2[(k + 1)y - x]. \tag{30}$$

Clearly, the minimum solution is obtained from (30), which is

$y - a(k + 1)y - x = 1, (k + 1)y - x = 2 \Longrightarrow y = 2a + 1$ (and $x = (k + 1)(2a + 1) - 2$).

Hence, in this case, the minimum m in (12) is $m = qy = q(2a + 1)$.

Case 3 : When p is of the form $p = 5a + 3$ for some integer $a \geq 2$.

In this case, from (27) and (28), we get respectively

$$1 = [(k+1)y - x](5a+3) - 5y = 3[(k+1)y - x] - 5[y - a(k+1)y - x], \tag{31}$$

$$1 = 5y - [(k+1)y - x](5a+3) = 5[y - a(k+1)y - x] - 3[(k+1)y - x]. \tag{32}$$

The minimum solution is obtained from (31) as follows :

$y - a(k+1)y - x = 1, (k+1)y - x = 2 \Longrightarrow y = 2a+1$ (and $x = (k+1)(2a+1) - 2$).

Hence, in this case, the minimum m in (12) is $m = qy - 1 = q(2a+1) - 1$.

Case 4 : When p is of the form $p = 5a+4$ for some integer $a \geq 2$.

In this case, 5 divides $p+1$. Then, the minimum solution is obtained from (28), which is $y = \frac{p+1}{5}$ (and $x = (k+1)y - 1$). Therefore, the minimum m in (12) is $m = qy = \frac{q(p+1)}{5}$.

Corollary 2.11. Let p and q be two primes with $q > p \geq 13$. Let $q = kp + 6$ for some integer $k \geq 1$. Then,

$$Z(pq) = \begin{cases} \frac{q(p-1)}{6}, & \text{if } 6|(p-1); \\ \frac{q(p+1)}{6} - 1, & \text{if } 6|(p+1). \end{cases}$$

Proof. From (2.1) and (2.2) with $\ell = 6$, we have respectively

$$6y - (x - ky)p = 1, \tag{33}$$

$$(x - ky)p - 6y = 1. \tag{34}$$

Now, for any prime $p \geq 13$, exactly one of the following two cases can occur : Either $p-1$ is divisible by 6, or $p+1$ is divisible by 6. We thus consider the two possibilities separately below :

Case 1 : When 6 divides $p-1$.

In this case, the minimum solution, obtained from (34), is $y = \frac{p-1}{6}$ (and $x = ky + 1$). Therefore, the minimum m in (12) is $m = qy = \frac{q(p-1)}{6}$.

Case 2 : When 6 divides $p+1$.

In this case, (33) gives the minimum solution, which is $y = \frac{p+1}{6}$ (and $x = ky+1$). Therefore, the minimum m in (12) is $m = qy - 1 = \frac{q(p+1)}{6} - 1$.

Corollary 2.12. Let p and q be two primes with $q > p \geq 13$. Let $q = (k+1)p - 6$ for some integer $k \geq 1$. Then,

$$Z(pq) = \begin{cases} \frac{q(p+1)}{6}, & \text{if } 6|(p+1); \\ \frac{q(p-1)}{6} - 1, & \text{if } 6|(p-1). \end{cases}$$

Proof. From (2.1) and (2.2) with $\ell = p - 6$, we have respectively

$$[(k+1)y - x]p - 6y = 1, \tag{35}$$

$$6y - [(k+1)y - x]p = 1. \tag{36}$$

We now consider the following two cases which are the only possibilities (as noted in the proof of Corollary 2.11) :

Case 1 : When 6 divides $p+1$.

In this case, the minimum solution, obtained from (36), is $y = \frac{p+1}{6}$ (and $x = (k+1)y - 1$). Therefore, the minimum m in (12) is $m = qy = \frac{q(p+1)}{6}$.

Case 2 : When 6 divides $p - 1$.

Here, the minimum solution is obtained from (35), which is $y = \frac{p-1}{6}$ (and $x = (k+1)y - 1$). Therefore, the minimum m in (12) is $m = qy = \frac{q(p-1)}{6} - 1$.

Corollary 2.13. Let p and q be two primes with $q > p \geq 13$. Let $q = kp + 7$ for some integer $k \geq 1$. Then,

$$Z(pq) = \begin{cases} \frac{q(p-1)}{7}, & \text{if } 7|(p-1); \\ q(3a+1) - 1, & \text{if } p = 7a+2; \\ q(2a+1) - 1, & \text{if } p = 7a+3; \\ q(2a+1), & \text{if } p = 7a+4; \\ q(3a+2), & \text{if } p = 7a+5; \\ \frac{q(p+1)}{7} - 1, & \text{if } 7|(p+1). \end{cases}$$

Proof. From (2.1) and (2.2) with $\ell = 7$, we have respectively

$$7y - (x - ky)p = 1, \tag{37}$$

$$(x - ky)p - 7y = 1. \tag{38}$$

Now, for any prime $p \geq 11$, exactly one of the following six cases occur :

Case 1 : When p is of the form $p = 7a + 1$ for some integer $a \geq 2$.
In this case, 7 divides $p - 1$. Then, the minimum solution is obtained from (38), which is $y = \frac{p-1}{7}$ (and $x - ky = 1$).
Therefore, the minimum m in (12) is $m = qy = \frac{q(p-1)}{7}$.

Case 2 : When p is of the form $p = 7a + 2$ for some integer $a \geq 2$.
In this case, from (37) and (38), we get respectively

$$1 = 7y - (x - ky)(7a + 2) = 7[y - (x - ky)a] - 2(x - ky), \tag{39}$$

$$1 = (x - ky)(7a + 2) - 7y = 2(x - ky) - 7[y - (x - ky)a]. \tag{40}$$

Clearly, the minimum solution is obtained from (39), which is

$y - (x - ky)a = 1, x - ky = 3 \Longrightarrow y = 3a + 1$ (and $x = k(3a + 1) + 3$).
Hence, in this case, the minimum m in (12) is $m = qy - 1 = q(3a + 1) - 1$.

Case 3 : When p is of the form $p = 7a + 3$ for some integer $a \geq 2$. Here, from (37) and (38),

$$1 = 7y - (x - ky)(7a + 3) = 7[y - (x - ky)a] - 3(x - ky), \tag{41}$$

$$1 = (x - ky)(7a + 3) - 7y = 3(x - ky) - 7[y - (x - ky)a]. \tag{42}$$

The minimum solution is obtained from (41) as follows:

$y - (x - ky)a = 1, x - ky = 2 \Longrightarrow y = 2a + 1$(and $x = k(2a + 1) + 2$).
Hence, in this case, the minimum m in (12) is $m = qy - 1 = q(2a + 1) - 1$.

Case 4 : When p is of the form $p = 7a + 4$ for some integer $a \geq 2$. In this case, from (37) and (38), we get respectively

$$1 = 7y - (x - ky)(7a + 4) = 7[y - (x - ky)a] - 4(x - ky), \tag{43}$$

$$1 = (x - ky)(7a + 4) - 7y = 4(x - ky) - 7[y - (x - ky)a]. \tag{44}$$

Clearly, the minimum solution is obtained from (44), which is

$y - (x - ky)a = 1, x - ky = 2 \Longrightarrow y = 2a + 1$(and $x = k(2a + 1) + 2$).

Hence, in this case, the minimum m in (12) is $m = qy - 1 = q(2a + 1) + 1$.

Case 5 : When p is of the form $p = 7a + 5$ for some integer $a \geq 2$. From (37) and (38), we have

$$1 = 7y - (x - ky)(7a + 5) = 7[y - (x - ky)a] - 5(x - ky), \tag{45}$$

$$1 = (x - ky)(7a + 5) - 7y = 5(x - ky) - 7[y - (x - ky)a]. \tag{46}$$

The minimum solution is obtained from (46), which is

$y - (x - ky)a = 2, x - ky = 3 \Longrightarrow y = 3a + 2$(and $x = k(3a + 2) + 3$).

Hence, in this case, the minimum m in (12) is $m = qy - 1 = q(3a + 2)$.

Case 6 : When p is of the form $p = 7a + 6$ for some integer $a \geq 2$. In this case, 7 divides $p + 1$. Then, the minimum solution is obtained from (37), which is $y = \frac{p+1}{7}$ (and $x - ky = 1$). Therefore, the minimum m in (12) is $m = qy - 1 = \frac{q(p+1)}{7} - 1$.

Corollary 2.14. Let p and q be two primes with $q > p \geq 13$. Let $q = (k + 1)p - 7$ for some integer $k \geq 1$.

Then,

$$Z(pq) = \begin{cases} \frac{q(p-1)}{7}, & \text{if } 7|(p-1); \\ q(3a+1), & \text{if } p = 7a+2; \\ q(2a+1), & \text{if } p = 7a+3; \\ q(2a+1)-1, & \text{if } p = 7a+4; \\ q(3a+2)-1, & \text{if } p = 7a+5; \\ \frac{q(p+1)}{7}, & \text{if } 7|(p+1). \end{cases}$$

Proof. From (2.1) and (2.2) with $\ell = 7$, we have respectively

$$[(k + 1)y - x]p - 7y = 1, \tag{47}$$

$$7y - [(k + 1)y - x]p = 1. \tag{48}$$

We now consider the following six possibilities:

Case 1 : When p is of the form $p = 7a + 1$ for some integer $a \geq 2$. In this case, 7 divides $p-1$. Then, the minimum solution is obtained from (47), which is $y = \frac{p-1}{7}$ (and$x = (k+1)y-1$). Therefore, the minimum m in (12) is $m = qy - 1 = \frac{q(p-1)}{7} - 1$.

Case 2 : When p is of the form $p = 7a + 2$ for some integer $a \geq 2$.

In this case, from (47) and (48), we get respectively

$$1 = [(k + 1)y - x](7a + 2) - 7y = 2[(k + 1)y - x] - 7[y - a\{(k + 1)y - x\}], \tag{49}$$

$$1 = 7y - [(k+1)y - x](7a+2) = 7[y - a\{(k+1)y - x\}] - 2[(k+1)y - x]. \tag{50}$$

Clearly, the minimum solution is obtained from (50), which is

$y - a\{(k+1)y - x\} = 1, (k+1)y - x = 3 \Longrightarrow y = 3a+1$(and $x = (k+1)(3a+1) - 3$).
Hence, in this case, the minimum m in (12) is $m = qy = q(3a+1)$.

Case 3 : When p is of the form $p = 7a+3$ for some integer $a \geq 2$.
Here, from (47) and (48),

$$1 = [(k+1)y - x](7a+3) - 7y = 3[(k+1)y - x] - 7[y - a\{(k+1)y - x\}], \tag{51}$$

$$1 = 7y - [(k+1)y - x](7a+3) = 7[y - a\{(k+1)y - x\}] - 3[(k+1)y - x]. \tag{52}$$

Then, (52) gives the minimum solution, which is:

$y - a\{(k+1)y - x\} = 1, (k+1)y - x = 2 \Longrightarrow y = 2a+1$(and $x = (k+1)(2a+1) - 2$).
Hence, in this case, the minimum m in (12) is $m = qy = q(2a+1)$.

Case 4 : When p is of the form $p = 7a+4$ for some integer $a \geq 2$.
Here, from (47) and (48),

$$1 = [(k+1)y - x](7a+4) - 7y = 4[(k+1)y - x] - 7[y - a\{(k+1)y - x\}], \tag{53}$$

$$1 = 7y - [(k+1)y - x](7a+4) = 7[y - a\{(k+1)y - x\}] - 4[(k+1)y - x]. \tag{54}$$

Clearly, the minimum solution is obtained from (53) as follows:

$y - a\{(k+1)y - x\} = 1, (k+1)y - x = 2 \Longrightarrow y = 2a+1$(and $x = (k+1)(2a+1) - 2$).
Hence, in this case, the minimum m in (12) is $m = qy - 1 = q(2a+1) - 1$.

Case 5 : When p is of the form $p = 7a+5$ for some integer $a \geq 2$.
In this case, from (47) and (48), we get respectively

$$1 = [(k+1)y - x](7a+5) - 7y = 5[(k+1)y - x] - 7[y - a\{(k+1)y - x\}], \tag{55}$$

$$1 = 7y - [(k+1)y - x](7a+5) = 7[y - a\{(k+1)y - x\}] - 5[(k+1)y - x]. \tag{56}$$

Then, (55) gives the following minimum solution:

$y - a\{(k+1)y - x\} = 2, (k+1)y - x = 3 \Longrightarrow y = 3a+2$ (and $x = (k+1)(3a+2) - 3$).
Hence, in this case, the minimum m in (12) is $m = qy - 1 = q(3a+2) - 1$.

Case 6 : When p is of the form $p = 7a+6$ for some integer $a \geq 2$. In this case, 7 divides $p+1$. Then, the minimum solution is obtained from (48), which is $y = \frac{p+1}{7}$ (and $x = (k+1)y - 1$). Therefore, the minimum m in (12) is $m = qy = \frac{q(p+1)}{7}$.

Corollary 2.15. Let p and q be two primes with $q > p \geq 13$. Let $q = kp + 8$ for some integer $k \geq 1$.

Then,

$$Z(pq) = \begin{cases} \frac{q(p-1)}{8}, & \text{if } 8|(p-1); \\ q(3a+1), & \text{if } p = 8a+3; \\ q(3a+2) - 1, & \text{if } p = 8a+5; \\ \frac{q(p+1)}{8} - 1, & \text{if } 8|(p+1). \end{cases}$$

Proof. From (2.1) and (2.2) with $\ell = 8$, we have respectively

$$8y - (x - ky)p = 1, \tag{57}$$

$$(x - ky)p - 8y = 1. \tag{58}$$

Now, for any prim $p \geq 13$, exactly one of the following four cases occur:

Case 1 : When p is of the form $p = 8a + 1$ for some integer $a \geq 2$.
In this case, 8 divides $p - 1$. Then, the minimum solution is obtained from (58), which is $y = \frac{p-1}{8}$ (and$x - ky = 1$).
Therefore, the minimum m in (12) is $m = qy = \frac{q(p-1)}{8}$.

Case 2 : When p is of the form $p = 8a + 3$ for some integer $a \geq 2$.
In this case, from (57) and (58), we get respectively

$$1 = 8y - (x - ky)(8a + 3) = 8[y - (x - ky)a] - 3(x - ky), \tag{59}$$

$$1 = (x - ky)(8a + 3) - 8y = 3(x - ky) - 8[y - (x - ky)a]. \tag{60}$$

Clearly, the minimum solution is obtained from (60), which is

$y - (x - ky)a = 1, x - ky = 3 \Longrightarrow y = 3a + 1$ (and $x = k(3a + 1) + 3$).
Hence, in this case, the minimum m in (12) is $m = qy = q(3a + 1)$.

Case 3 : When p is of the form $p = 8a + 5$ for some integer $a \geq 2$.
From (57) and (58), We get

$$1 = 8y - (x - ky)(8a + 5) = 8[y - (x - ky)a] - 5(x - ky), \tag{61}$$

$$1 = (x - ky)(8a + 5) - 8y = 5(x - ky) - 8[y - (x - ky)a]. \tag{62}$$

The minimum solution is obtained from (61) as follows:

$y - (x - ky)a = 2, x - ky = 3 \Longrightarrow y = 3a + 2$ (and $x = k(3a + 2) + 3$).
Hence, in this case, the minimum m in (12) is $m = qy - 1 = q(3a + 2) - 1$.

Case 4 : When p is of the form $p = 8a + 7$ for some integer $a \geq 2$.
In this case, 8 divides $p - 1$. Then, the minimum solution is obtained from (57), which is $y = \frac{p+1}{8}$ (and$x - ky = 1$). Therefore, the minimum m in (12) is $m = qy - 1 = \frac{q(p+1)}{8} - 1$.

Corollary 2.16. Let p and q be two primes with $q > p \geq 13$. Let $q = (k + 1)p - 8$ for some integer $k \geq 1$.

Then,

$$Z(pq) = \begin{cases} \frac{q(p-1)}{8}, & \text{if } 8|(p-1); \\ q(3a+1) - 1, & \text{if } p = 8a + 3; \\ q(3a+2), & \text{if } p = 8a + 5; \\ \frac{q(p+1)}{8}, & \text{if } 8|(p+1). \end{cases}$$

Proof. From (2.1) and (2.2) with $\ell = p - 8$, we have respectively

$$[(k + 1)y - x]p - 8y = 1, \tag{63}$$

$$8y - [(k + 1)y - x]p = 1. \tag{64}$$

We now consider the four possibilities that may arise:

Case 1 : When p is of the form $p = 8a + 1$ for some integer $a \geq 2$.

In this case, 8 divides $p - 1$. Then, the minimum solution is obtained from (63), which is $y = \frac{p-1}{8}$ (and $x = (k+1)y - 1$). Therefore, the minimum m in (12) is $m = qy - 1 = \frac{q(p-1)}{8} - 1$.

Case 2 : When p is of the form $p = 8a + 3$ for some integer $a \geq 2$.

In this case, from (63) and (64), we get respectively

$$1 = [(k+1)y - x](8a+3) - 8y = 2[(k+1)y - x] - 8[y - a\{(k+1)y - x\}], \tag{65}$$

$$1 = 8y - [(k+1)y - x](8a+3) = 8[y - a\{(k+1)y - x\}] - 3[(k+1)y - x]. \tag{66}$$

Clearly, the minimum solution is obtained from (65), which is

$$y - a\{(k+1)y - x\} = 1, (k+1)y - x = 3 \Longrightarrow y = 3a + 1(\text{and} x = (k+1)(3a+1) - 3).$$

Hence, in this case, the minimum m in (12) is $m = qy = q(3a+1) - 1$.

Case 3 : When p is of the form $p = 8a + 5$ for some integer $a \geq 1$.

In this case, from (63) and (64), we get respectively

$$1 = [(k+1)y - x](8a+5) - 8y = 5[(k+1)y - x] - 8[y - a\{(k+1)y - x\}], \tag{67}$$

$$1 = 8y - [(k+1)y - x](8a+5) = 8[y - a\{(k+1)y - x\}] - 5[(k+1)y - x]. \tag{68}$$

The minimum solution is obtained from (68) as follows:

$$y - a\{(k+1)y - x\} = 2, (k+1)y - x = 3 \Longrightarrow y = 3a + 2(\text{and} x = (k+1)(3a+2) - 3).$$

Hence, in this case, the minimum m in (12) is $m = qy = q(3a+2)$.

Case 4 : When p is of the form $p = 8a + 7$ for some integer $a \geq 2$.

In this case, 8 divides $p + 1$. Then, the minimum solution is obtained from (64), which is $y = \frac{p+1}{8}$ (and $x = (k+1)y - 1$). Therefore, the minimum m in (12) is $m = qy = \frac{q(p+1)}{8}$.

We now consider the case when n is a composite number. Let

$Z(n) = m_0$ for some integer $m_0 \geq 1$. Then, n divides $\frac{m_0(m_0+1)}{2}$.

We now consider the following two cases that may arise :

Case 1 : m_0 is even (so that $m_0 + 1$ is odd).

(1) Let n be even. In this case, n does not divide $\frac{m_0}{2}$, for otherwise,

$$\frac{n|m_0}{2} \Longrightarrow \frac{n|m_0(m_0+1)}{2} \Longrightarrow Z(n) \leq (m_0 - 1).$$

(2) Let n be odd. In such a case, n does not divide m_0.

Case 2 : m_0 is odd (so that $m_0 + 1$ is even).

(1) Let n be even. Then, n does not divide m_0.

(2) Let n be odd. Here, n does not divide m_0, for

$$n|m_0 \Longrightarrow \frac{n|m_0(m_0-1)}{2} \Longrightarrow Z(n) \leq (m_0 - 1).$$

Thus, if n is a composite number, n does not divide m_0.

Now let

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_i^{\alpha_i} p_{i+1}^{\alpha_{i+1}} \cdots p_s^{\alpha_s}$$

be the representation of n in terms of its distinct prime factors $p_1, p_2, \cdots p_i, p_{i+1}, \cdots p_s$, not necessarily ordered. Then, one of m_0 and $m_0 + 1$ is of the form

$$2^{\beta} p_1^{\beta_1} p_2^{\beta_2} \cdots p_i^{\beta_i} q_{i+1}^{\beta_{i+1}} \cdots q_s^{\beta_s}$$

for some $1 \leq i < s$; $\beta_j \geq \alpha_j$ for $1 \leq j < i$, and the other one is of the form

$$p_{i+1}^{\gamma_{i+1}} \cdots p_s^{\gamma_s}, r_{s+1}^{\gamma_{s+1}} \cdots r_u^{\gamma_u} \gamma_j \geq \alpha_j$$

for $i + 1 \leq j < s$; where $q_{i+1}, \cdots q_s$ and $r_{s+1}, \cdots r_u$ are all distinct primes, not necessarily ordered.

§3. Some Observations

Some observations about the Pseudo-Smarandache Function are given below :

Remark 3.1. Kashihara raised the following questions (see Problem 7 in [1]) :
(1) Is there any integer n such that $Z(n) > Z(n+1) > Z(n+2) > Z(n+3)$?
(2) Is there any integer n such that $Z(n) < Z(n+1) < Z(n+2) < Z(n+3)$?
The following examples answer the questions in the affirmative:

(1) $Z(256) = 511 > 256 = Z(257) > Z(258) = 128 > 111 = Z(259) > Z(260) = 39,$

(2) $Z(159) = 53 < 64 = Z(160) < Z(161) = 69 < 80 = Z(162) < Z(163) = 162.$

These examples show that even five consecutive increasing or decreasing terms are available in the sequence $\{Z(n)\}$.

Remark 3.2 Kashihara raises the following question (see Problem 5 in [1]) : Given any integer $m_0 \geq 1$, how many n are there such that $Z(n) = m_0$?
Given any integer $m_0 \backslash 3$, let

$$Z^{-1}(m_0) = \{n : n \in N, Z(n) = m_0\}, \tag{2.3}$$

with

$$Z^{-1}(1) = \{1\}, Z^{-1}(2) = \{3\}. \tag{2.4}$$

Thus, for example, $Z^{-1}(8) = \{8, 12, 18, 36\}$.
By Lemma 2.1,

$$n_{\max} \equiv \frac{m_0(m_0+1)}{2} \in Z^{-1}(m_0).$$

This shows that the set $Z^{-1}(m_0)$ is non-empty; moreover, $n_{\max}$ is the biggest element of $Z^{-1}(m_0)$, so that $Z^{-1}(m_0)$ is also bounded. Clearly, $n \in Z^{-1}(m_0)$ only if n divides $f(m_0) \equiv$

$m_0(m_0+1)/2$. This is a necessary condition, but is not sufficient. For example, $4|36 \equiv f(8)$ but $4 \notin Z^{-1}(8)$. The reason is that $Z(n)$ is not bijective. Let

$$Z^{-1} \equiv \sum_{m=1}^{\infty} Z^{-1}(m)$$

Let $n \in Z^{-1}$. Then, there is one and only one mo such that $n \in Z^{-1}(m_0)$, that is, there is one and only one mo such that $Z(n) = m_0$.

However, we have the following result whose proof is almost trivial : $n \in Z^{-1}(m_0)(n \neq 1, 3)$ if and only if the following two conditions are satisfied

(1) n divides $m_0(m_0+1)/2$,

(2) n does not divide $m(m+1)/2$ for any m with $3 \leq m \leq m_0 - 1$.

Since $4|28 \equiv f(7)$, it therefore follows that $4 \notin Z^{-1}(8)$.

Given any integer $m_0 \geq 1$, let $C(m_0)$ be the number of integers n such that $Z(n) = m_0$, that is, $C(m_0)$ denotes the number of elements of $Z^{-1}(m_0)$. Then,

$$1 \leq C(m_0) \leq d(m_0(m_0+1)/2) - 1 \quad for m_0 \geq 3; \quad C(1) = 1, \quad C(2) = 2,$$

where, for any integer n, $d(n)$ denotes the number of divisors of n including 1 and n. Now, let $p \geq 3$ be a prime. Since, by Lemma 1.2, $Z(p) = p-1$, we see that $p \in Z^{-1}(p-1)$ for all $p \geq 3$. Let $n \in Z^{-1}(p-1)$. Then, n divides $p(p-1)/2$. This shows that n must divide p, for otherwise

$$n|\frac{p-1}{2} \Rightarrow n|\frac{(p-1)(p-2)}{2} \Rightarrow Z(n) \leq p-2,$$

contradicting the assumption. Thus, any element of $Z^{-1}(p-1)$ is a multiple of p. In particular, p is the minimum element of $Z^{-1}(p-1)$. Thus, if $p \geq 5$ is a prime, then $Z^{-1}(p-1)$ contains at least two elements, namely, p and $p(p-1)/2$. Next, let p be a prime factor of $m_0(m_0+1)/2$. Since, by Lemma 1.2, $Z(p) = p-1$, we see that $p \in Z^{-1}(m_0)$ if and only if $p - 1 \geq m_0$, that is, if and only if $p \geq m_0 + 1$.

Remark 3.3. Ibstedt[2] provides a table of values of $Z(n)$ for $1 \leq n \leq 1000$. A closer look at these values reveal some facts about the values of $Z(n)$. These observations are given in the conjectures below, followed by discussions in each case.

Conjecture 1. $Z(n) = 2n - 1$ if and only if $n = 2^k$ for some integer $k \geq 0$.

Let, for some integer $n \geq 1$,

$$Z(n) = m_0, \text{ where } m_0 = 2n - 1.$$

Note that the conjecture is true for $n = 1$ (with $k = 0$). Also, note that n must be composite.

Now, since $m_0 = 2n - 1$, and since $n|\frac{m_0(m_0+1)}{2}$, it follows that

n does not divide m_0, and $n|\frac{m_0+1}{2}$; moreover, by virtue of the definition of $Z(n)$, n does not divide m_0, and $n|\frac{m+1}{2}$ for all $1 \leq m \leq m_0 - 1$.

Let

$$Z(2n) = m_1.$$

We want to show that $m_1 = 2m_o + 1$. Since $n|\frac{m_0+1}{2}$, it follows that $2n|\frac{2(m_0+1)}{2} = \frac{(2m_0+1)+1}{2}$; moreover, $2n$ does not divide

$$\frac{2(m+1)}{2} = \frac{(2m+1)+1}{2}$$

for all $1 \leq m \leq m_0 - 1$.
Thus,

$$m_1 = 2m_0 + 1 = 2(2n-1) + 1 = 2^2 n - 1.$$

All these show that

$$Z(n) = 2n - 1 \Rightarrow Z(2n) = 2^2 n - 1.$$

Continuing this argument, we see that

$$Z(n) = 2n - 1 \Rightarrow Z(2^k n) = 2^{k+1} n - 1.$$

Since $Z(1) = 1$, it then follows that $Z(2^k) = 2^{k+1} - 1$.

Conjecture 2. $Z(n) = n-1$ if and only if $n = p^k$ for some prime $p \geq 3$ and integer $k \geq 1$.
Let, for some integer $n \geq 2$,

$$Z(n) = m_0, \text{ where } m_0 = n - 1.$$

Then, $2|m_0$ and $n|(m_0 + 1)$; moreover, n does not divide $m + 1$ for any $1 \leq m \leq m_0 - 1$.
Let

$$Z(n^2) = m_1.$$

Since $n|(m_0 + 1)$, it follows that

$$n^2|(m_0 + 1)^2 = (m_0^2 + 2m_0) + 1;$$

moreover, n^2 does not divide$|(m + 1)^2 = (m^2 + 2m) + 1$ for all $1 \leq m \leq m_0 - 1$.
Thus,

$$m_1 = m_0^2 + 2m_0 = (n-1)^2 + 2(n-1) = n^2 - 1,$$

so that(since $2|m_0 \Rightarrow 2|m_1$)

$$Z(n) = n - 1 \Rightarrow Z(n^2) = n^2 - 1.$$

Continuing this argument, we see that

$$Z(n) = n - 1 \Rightarrow Z(n^{2k}) = n^{2k} - 1.$$

Next, let

$$Z(n^{2k+1}) = m_2 \text{ for some integer } k \geq 1.$$

Since $n|(m_0 + 1)$, it follows that

$$n^{2k+1}|(m_0 + 1)^{2k+1} = [(m_0 + 1)^{2k+1} - 1] + 1;$$

moreover,
n^{2k+1} does not divide

$$|(m + 1)^{2k+1} = [(m + 1)^{2k+1} - 1] + 1$$

for all $1 \leq m \leq m_0 - 1$. Thus,

$$m_2 = (m_0 + 1)^{2k+1} - 1 = n^{2k+1} - 1,$$

so that (since $2|m_0 \Rightarrow 2|m_2$)

$$Z(n) = n - 1 \Rightarrow Z(n^{2k+1}) = n^{2k+1} - 1.$$

All these show that

$$Z(n) = n - 1 \Rightarrow Z(n^k) = n^k - 1.$$

Finally, since $Z(p) = p - 1$ for any prime $p \geq 3$, it follows that $Z(p^k) = p^k - 1$.

Conjecture 3. If n is not of the form 2^k for some integer $k \geq 0$, then $Z(n) < n$. First note that, we can exclude the possibility that $Z(n) = n$, because

$$n|\frac{n(n+1)}{2} \Rightarrow n|\frac{n(n-1)}{2} \Rightarrow Z(n) \leq n - 1.$$

So, let

$$Z(n) = m_0 \text{ with } m_0 > n.$$

Note that, n must be a composite number, not of the form p^k ($p \geq 3$ is prime, $k \geq 0$). Let

$$m_0 = an + b \text{ for some integers } a \geq 1, 1 \leq b \leq n1.$$

Then,

$$m_0(m_0 + 1) = (an + b)(an + b + 1) = n(a^2n + 2ab + a) + b(b + 1).$$

Therefore,

$$n|m_0(m_0 + 1) \text{ if and only if } b + 1 = n.$$

But, by Conjecture 1, $b + 1 = n$ leads to the case when n is of the form 2^k.

Remark 3.4. Kashihara proposes (see Problem 4(a) in [1]) to find all the values of n such that $Z(n) = Z(n + 1)$. In this connection, we make the following conjecture :

Conjecture 4. For any integer $n \geq 1$, $Z(n) \neq Z(n + 1)$. Let

$$Z(n) = Z(n + 1) = m_0 \text{ for some } n \in N, m_0 \geq 1. \tag{69}$$

Then, neither n nor $n + 1$ is a prime.

To prove this, let $n = p$, where p is a prime. Then, by Lemma 1.2, $Z(n) = Z(p) = p - 1$.

$$n + 1 = p + 1 \text{ does not divide } \frac{p(p-1)}{2} \Rightarrow Z(n + 1) \neq p - 1 = Z(n).$$

Similarly, it can be shown that $n + 1$ is not a prime. Thus, both n and $n + 1$ are composite numbers.

From (68), we see that both n and $n + 1$ divide $m_0(m_0 + 1)/2$. Let

$$\frac{m_0(m_0 + 1)}{2} = an \text{ for some integer } a \geq 1.$$

Since $n + 1$ divides $m_0(m_0 + 1)$ and since $n + 1$ does not divide n, it follows that $n + 1$ must divide a. So, let

$$a = b(n + 1) \text{ for some integer } b \geq 1.$$

Then,

$$\frac{m_0(m_0+1)}{2} = abn(n+1),$$

which shows that

$$n(n+1) \text{ must divide } \frac{m_0(m_0+1)}{2}. \tag{70}$$

From (69), we see that

$$Z(n(n+1)) \leq m_0,$$

which, together with Lemma 1.5 (that $Z(n(n+1)) \geq Z(n)$), gives

$$Z(n(n+1)) = m_0. \tag{71}$$

From (70), we see that

$$n(n+1)\frac{m_0(m_0+1)}{2} \Rightarrow \frac{n(n+1)}{2}|\frac{m_0(m_0+1)}{2} \Rightarrow Z(\frac{n(n+1)}{2}) \leq m_0.$$

Thus, by virtue of Lemma 2.1, $Z(\frac{n(n+1)}{2}) = n \leq m_0 = Z(n)$. It can easily be verified that neither n nor $n+1$ can be of the form $2k$. Thus, if Conjecture 3 is true then Conjecture 4 is also true.

Remark 3.5. An integer $n > 0$ is called f-perfect if

$$n = \sum_{i=1}^{k} f(d_i),$$

where $d_1 \equiv 1, d_2, \ldots, d_k$ are the proper divisors of n, and f is an arithmetical function. In particular, n is Pseudo-Smarandache perfect if

$$n = \sum_{i=1}^{k} Z(d_i).$$

In [4], Ashbacher reports that the only Pseudo-Smarandache perfect numbers less than $1,000,000$ are $n = 4, 6, 471544$. However, since $n = 471544$ is of the form $n = 8p$ with $p = 58943$, its only perfect divisors are $1, 2, 4, 8, p, 2p$ and $4p$. Since $8|(p+1) = 58944$, it follows from Lemma 1.2, Theorem 2.1 and Theorem 2.7 that

$$Z(p) = p - 1, \quad Z(2p) = p, \quad Z(4p) = p,$$

so that

$$n = 471544 > \sum_{i=1}^{k} Z(d_i),$$

so that $n = 471544$ is not Pseudo-Smarandache perfect.

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Primes in the Smarandache deconstructive sequence

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Abstract In this article, we present the results of investigation of first 10000 terms of Smarandache Deconstructive Sequence and report some new primes and other results found from the sequence.

Keywords The Smarandache deconstructive sequence, initial digits, trailing digits.

Introduction

The Smarandache Deconstructive sequence of integers [1] is constructed by sequentially repeating the digits 1 to 9 as follows:

1, 23, 456, 7891, 23456, 789123, $\cdots\cdots$

Kashihara [2] asked: How many primes are there in the sequence. Ashbacher [3] explored this question and raised some more questions, which were studied and answered by Henry Ibstedt [4].

Let us call the sequence mentioned above as Smarandache Deconstructive sequence of the first kind (SDS-I) because a similar Deconstructive sequence can be constructed by sequentially repeating the digits 0 to 9 as follows [4].

0, 12, 345, 6789, 01234, 567890, $\cdots\cdots$

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0, 12, 345, 6789, 01234, 567890, $\cdots\cdots$

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In this paper, we report the primes found in both the sequence after checking the first 10000 terms of both these sequence.

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Primes in the Smarandache Deconstructive Sequence of first kind:

The following 13 primes in the Smarandache Deconstructive sequence of first kind have been reported earlier [5] [6].

$23,\ 4567891,\ 23456789, \cdots\cdots$

These are 2, 7, 8, 10, 17, 20, 25, 28, 31, 38, 61, 62 and 355-th term of the sequence and are given in Table-1 below:

Table 1

Term	Prime
2	23
7	4567891
8	23456789
10	1234567891
17	23456789123456789
20	23456789123456789123
25	4567891234567891234567891
28	1234567891234567891234567891
31	7891234567891234567891234567891
38	23456789123456789123456789123456789123
61	4567891234567891234567891234567891234567891234567891234567891
62	23456789123456789123456789123456789123456789123456789123456789
355	$789(123456789)_{39}1$

Note that $(123456789)_{39}$ means 123456789 repeated 39 times. On further computation up to 10000 terms of the sequence, we have noted following 3 more primes, namely the term 4690, 4772 and 8162 of the sequence. Since the primality of the term 4690, 4772 and 8162 have not been certified, so these can be treated as probable primes. These are:
$(123456789)_{521}1$,
$23456789(123456789)_{529}123$,
$23456789(123456789)_{906}$.

It may be noted that though there are 12 primes in the first 62 terms of the sequence but only 16 primes in the first 10000 terms of the sequence. So the percentage of primes is reducing significantly which is in accordance with prime number theorem, according to which, the probability that a random chosen number of size n is prime decreases as $\frac{1}{d}$ (where d is the number of digits of n).

Observations on the Smarandache Deconstructive Sequence of first kind:

From the term of this sequence, it is seen that the trailing digit (units digit) repeats the pattern.

1, 3, 6, 1, 6, 3, 1, 9, 9; · · · · · ·

Interestingly this sequence is the same as the sequence of digital root of triangular numbers.

Similarly initial digit of the element of SDS-I repeats the pattern

1, 2, 4, 7, 2, 7, 4, 2, 1; · · · · · ·

Table-2 below gives the possible combination of initial and trailing digits of any element of SDS-I.

Table 2

Trailing digits	Initial digits
1	1, 4, 7
3	2, 7
6	4, 2
9	2, 1

Since the trailing digits of the term of the SDS-I sequence can be 1, 3, 6 *or* 9, it is obvious that for an element to be prime, the only possible trailing digits are 1, 3 or 9. If trailing digit is 3, possible initial digits are 2 and 7, but if initial digit is 7 and trailing digit is 3, the number is divisible by 3. Similarly if trailing digit is 9 and initial digit is 1, the number is divisible by 3. The possible combinations of trailing and initial digits for a prime in the sequence are given in Table-3.

Table 3

Trailing digits	Initial digits
1	1, 4, 7
3	2
9	2

So there are 5 possibilities out of 9, as the pattern repeat for every 9 elements in the sequence. Out of 13 primes found, 7 ends in 1, 3 ends in 3, 3 end in 9 and primes corresponding to all 5 possibilities are found. The three probable primes found end in 1, 3 and 9 respectively.

It is thus clear that the term $3 + 9n$, $5 + 9n$, $6 + 9n$, $9n$ of the sequence are obviously composite and need not be checked for primality. Only the term $9n+1$, $9n+2$, $9n+4$, $9n+7$ and $9n + 8$ need to be checked for primality.

Conjecture 1. Every prime except 5 divides some element of the sequence.

It is noted that none of the element of the sequence end in 0 or 5. So 5 cannot be a factor of any terms of the sequence. It has been checked that every prime up to 3821 except 5 divides some element of the sequence up to 10000 terms, so it is quite reasonable to conjecture that: every prime except 5 divides some element of the sequence. Can this be proved?

Primes in the Smarandache Deconstructive Sequence of second kind:

On computation up to 10000 terms of the sequence, we have noted only 2 primes, namely the term 367 and 567 of the sequence. These are:
$(1234567890)_{36}1234567$,
$(1234567890)_{56}1234567$.

Observations on the Smarandache Deconstructive Sequence of second kind:

From the terms of this sequence, it is seen that the trailing digit (units digit) repeats the pattern.

0, 2, 5, 9, 4, 0, 7, 5, 4, 4, 5, 7, 0, 4, 9, 5, 2, 0, 9, 9; $\cdots\cdots$

Similarly initial digit of the element of SDS-II repeats the pattern

0, 1, 3, 6, 0, 5, 1, 8, 6, 5, 5, 6, 8, 1, 5, 0, 6, 3, 1, 0; $\cdots\cdots$

Table-4 below gives the possible combination of initial and trailing digits of any element of SDS-II

Table 4

Trailing digits	Initial digits
0	0, 5, 8, 3
2	1, 6
4	0, 5, 6, 1
5	3, 5, 8, 0
7	1, 6
9	1, 0, 5, 6

Since the trailing digits of the term of the SDS-II sequence can be 0, 2, 4, 5, 7 or 9, it is obvious that for an element to be prime, the only possible trailing digits are 7 or 9. If trailing digit is 7, possible initial digits are 1 or 6. Similarly if trailing digit is 9, possible initial digits are 0, 1, 5 or 6. If initial digit is 0, 1 or 6 and trailing digit is 9, the number is divisible by 3. So it cannot be prime. The possible combinations of trailing and initial digits for a prime in the sequence are given in Table-5.

Table 5

Trailing digits	Initial digits
7	1, 6
9	5

So there are 3 possibilities out of 20, as the pattern repeat for every 20 elements in the sequence. Both the primes in first 10000 terms of the sequence end in 7 and initial digit in both primes is 1. It remains to find a single prime corresponding to trailing digit 9 and also corresponding to trailing digit 7 with initial digit 6. The only terms needs to be checked for

primality are $7 + 20n$, $12 + 20n$ and $15 + 20n$. It is interesting to note that for every 20 terms of the sequence, only 3 needs to be checked for possible primes, whereas in SDS-I, for every 9 terms of the sequence, 5 terms needs to be checked for possible primes. This gives an indication that if there are n_1 possible primes in SDS-I, then in SDS-II, the number of possible primes $n_2 = n_1 * (\frac{3}{20}) * (\frac{9}{5}) = 0.27n_1$ This explains why the number of primes found in SDS-II is fewer as compared to number of primes found in SDS-I. The time required to search for primes in SDS-I is also correspondingly higher than the time required to search for primes in SDS-II.

Conjecture 2. Every prime divides some element of the sequence. It has been checked that every prime up to 2591 divides some element of the sequence up to 10000 terms, so it is again quite reasonable to conjecture that: every prime divides some element of the sequence. Can this be proved?

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Recursion formulae for Riemann Zeta function and Dirchlet series[1]

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Abstract In this paper, some recursion formulae of sums for the Riemann Zeta function and Dirchlet series are obtained through expanding several simple function on $[-\pi, \pi]$ or $[0, 2\pi]$ by using the Dirichlet theorem in Fourier series theory.

Keywords Riemann zeta function, Dirichlet series, recurrence formula, Fourier series.

§1. Introduction

It is well-known that the Riemann Zeta function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re(s) > 1 \tag{1}$$

and Dirichlet series

$$D(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \quad \Re(s) > 1 \tag{2}$$

play very important roles in Analytic Number Theory, and so on.

In 1734, Euler gave sum of the following Bernoulli series

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{3}$$

The formula (3) has been studied by many mathematicians and many proofs have been published, for example, see [2]. In 1748, Euler further gave the following general formula

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1} 2^{2k-1} \pi^{2k}}{(2k)!} B_{2k}, \tag{4}$$

where $B_{2k} (k = 1, 2, \ldots)$ denotes Bernoulli numbers, defined in [18,19] by

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \frac{t^k}{k!} B_k, \quad |t| < 2\pi.$$

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For other proofs concerning formula (4), please refer to the references in this paper, for example, [18] and [21]. In 1999, the paper [9] gave an elementary expression for $\zeta(2k)$: Let $n \in \mathbb{N}$, then

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = A_k \pi^{2k}, \tag{5}$$

where

$$\begin{aligned} A_k &= \frac{1}{3!}A_{k-1} - \frac{1}{5!}A_{k-2} + \cdots + (-1)^{k-2}\frac{1}{(2k-1)!}A_1 + (-1)^{k-1}\frac{k}{(2k+1)!} \\ &= (-1)^{k-1}\frac{k}{(2k+1)!} + \sum_{i=1}^{k-1}\frac{(-1)^{k-i-1}}{(2k-2i+1)!}A_i. \end{aligned} \tag{6}$$

It is still an open problem to prove irrationality of $\zeta(2k+1)$ for several centuries. Until 1978, R. Apéry, a French mathematician, proved that the number $\zeta(3)$ is irrational. But one can not generalize his proof to other cases. So, many mathematicians keep much interest in the evaluation of $\zeta(s)$ and sums of related series. For some examples, see [10,20,22].

The following formulae involving $\zeta(2k+1)$ were given by Ramanujan, see [22], as follows:

1. If $k > 1$ and $k \in \mathbb{N}$,

$$\alpha^k \left[\frac{1}{2}\zeta(1-2k) + \sum_{n=1}^{\infty}\frac{n^{2k-1}}{e^{2n\alpha}-1}\right] = (-\beta)^k \left[\frac{1}{2}\zeta(1-2k) + \sum_{n=1}^{\infty}\frac{n^{2k-1}}{e^{2n\beta}-1}\right], \tag{7}$$

2. if $k > 0$ and $k \in \mathbb{N}$,

$$\begin{aligned} 0 = {} & \frac{1}{(4\alpha)^k}\left[\frac{1}{2}\zeta(2k+1) + \sum_{n=1}^{\infty}\frac{1}{n^{2k+1}(e^{2n\alpha}-1)}\right] \\ & - \frac{1}{(-4\beta)^k}\left[\frac{1}{2}\zeta(2k+1) + \sum_{n=1}^{\infty}\frac{1}{n^{2k+1}(e^{2n\beta}-1)}\right] \\ & + {\sum_{j=0}^{\left[\frac{k+1}{2}\right]}}' \frac{(-1)^j \pi^{2j} B_{2j} B_{2k-2j+2}}{(2j)!(2K-2J+2)!}\left[\alpha^{k-2j+1} + (-\beta)^{k-2j+1}\right], \end{aligned} \tag{8}$$

where B_j is the j-th Bernoulli number, $\alpha > 0$ and $\beta > 0$ satisfy $\alpha\beta = \pi^2$, and $\sum'$ means that, when k is an odd number $2m-1$, the last term of the left hand side in (8) is taken as $\dfrac{(-1)^m \pi^{2m} B_{2m}^2}{(m!)^2}$.

In 1928, Hardy in [6] proved (7). In 1970, E. Grosswald in [3] proved (8). In 1970, E. Grosswald in [4] gave another expression of $\zeta(2k+1)$. In 1983, N.-Y. Zhang in [20] not only proved Ramanujan formulae (7) and (8), but also gave an explicit expression of $\zeta(2k+1)$ as follows:

1. If k is odd, then we have

$$\zeta(2k+1) = -2\psi_{-k}(\pi) - (2\pi)^{2k+1} {\sum_{j=0}^{\left[\frac{k+1}{2}\right]}}' \frac{(-1)^j \pi^{2j} B_{2j} B_{2k-2j+2}}{(2j)!(2k-2j+2)!}; \tag{9}$$

2. if k is even,

$$\zeta(2k+1) = -2\psi_{-k}(\pi) + \frac{2\pi}{k}\psi'_{-k}(\pi)\frac{(2\pi)^{2k+1}}{k}\sum_{j=0}^{\frac{k}{2}}\frac{(-1)^j\pi^{2j}B_{2j}B_{2k-2j+2}}{(2j)!(2k-2j+2)!}, \tag{10}$$

where $\psi_{-k}(\alpha) = \sum_{n=1}^{\infty}\frac{1}{n^{2k+1}(e^{2n\alpha}-1)}$, and $\psi'_{-k}(\alpha)$ is the derivative of $\psi_{-k}(\alpha)$ with respect to α.

There are a lot of literature on calculating of $\zeta(s)$, for example, see [2,p.435] and [18,pp.144-145; p.149; pp.150-151]. As a matter of fact, many other recent investigations and important results on the subject of the Riemannian Zeta function $\zeta(s)$ can be found in the papers [11,12,13,14,15,16,17] by H. M. Srivastava, and others. Furthermore, Chapter 4 entitled "Evaluations and Series Representations" of the book [15] contains a rather systematic presentation of much of these recent developments.

The aim of this paper is to obtain recursion formulae of sums for the Riemann Zeta function and Dirchlet series through expanding some simple function on $[-\pi,\pi]$ or $[0,2\pi]$ by using the Dirichlet theorem in Fourier series theory.

§2. Main results and proofs

Theorem 1. Let $\delta(s) \triangleq \sum_{n=1}^{\infty}\frac{1}{(2n-1)^s}$ and $\sigma(s) \triangleq \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^s}$ for $s>1$. Then we have for $k\in\mathbb{N}$

$$D(2k) = \frac{(-1)^{k+1}(4k-1)\pi^{2k}}{2\cdot(2k)!} + \sum_{j=1}^{k}\frac{(-1)^{k+j+1}4^j\pi^{2k-2j}}{(2k-2j)!}\zeta(2j), \tag{11}$$

$$D(2k) = \frac{(-1)^{k+1}\pi^{2k}}{4k\cdot(2k-2)!} + \sum_{j=1}^{k}\frac{(-1)^{k+j+1}\pi^{2k-2j}}{(2k-2j)!}\zeta(2j), \tag{12}$$

$$D(2k) = \frac{(-1)^{k+1}(4k-3)3^{2k-1}\pi^{2k}}{2\cdot(2k)!} + \sum_{j=1}^{k}\frac{(-1)^{k+j+1}4^j(3\pi)^{2k-2j}}{(2k-2j)!}\zeta(2j), \tag{13}$$

$$\zeta(2k) = \frac{(-1)^{k+1}(2k-1)2^{2k-2}\pi^{2k}}{(2k+1)!} + \sum_{j=1}^{k-1}\frac{(-1)^{k+j+1}(2\pi)^{2k-2j}}{(2k-2j+1)!}\zeta(2j), \tag{14}$$

$$\sigma(2k+1) = \frac{(-1)^k(4k+1)}{2\cdot(2k+1)!}\left(\frac{\pi}{2}\right)^{2k+1} + \sum_{j=1}^{k}\frac{(-1)^{k+j}}{(2k-2j+1)!}\left(\frac{\pi}{2}\right)^{2k-2j+1}\zeta(2j), \tag{15}$$

$$\zeta(2k) = \frac{(-1)^{k+1}k\pi^{2k}}{(2k+1)!} + \sum_{j=1}^{k-1}\frac{(-1)^{k+j+1}\pi^{2k-2j}}{(2k-2j+1)!}\zeta(2j), \tag{16}$$

$$\sigma(2k+1) = \frac{(-1)^{k+1}(4k-1)\pi^{2k+1}}{4\cdot(2k+1)!}\left(\frac{3}{2}\right)^{2k} + \sum_{j=1}^{k}\frac{(-1)^{k+j+1}}{(2k-2j+1)!}\left(\frac{3\pi}{2}\right)^{2k-2j+1}\zeta(2j), \tag{17}$$

$$\zeta(2k) = \frac{(-1)^{k+1}k(2\pi)^{2k}}{(2k+2)!} + \sum_{j=1}^{k-1}\frac{(-1)^{k+j+1}2^{2k-2j+1}\pi^{2k-2j}}{(2k-2j+2)!}\zeta(2j). \tag{18}$$

Proof. Define the function f by

$$f(x) = \frac{\pi - x}{2}, \quad x \in (0,\ 2\pi).$$

Easy computation reveals the Fourier series of f on $(0, 2\pi)$:

$$\frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{\sin nx}{n}, \quad x \in (0,\ 2\pi). \tag{19}$$

Integration term-by-term yields

$$-\frac{x^2}{2 \cdot 2!} + \frac{\pi x}{2} - \zeta(2) = -\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}, \quad x \in [0,\ 2\pi].$$

Clearly, if we integrate $2k-1$ times on each side of (19) from 0 to x, then we obtain

$$\begin{aligned} &-\frac{x^{2k}}{2 \cdot (2k)!} + \frac{\pi x^{2k-1}}{2 \cdot (2k-1)!} + \sum_{j=1}^{k} \frac{(-1)^j x^{2k-2j}}{(2k-2j)!} \zeta(2j) \\ &= (-1)^k \sum_{n=1}^{\infty} \frac{\cos nx}{n^{2k}}, x \in [0,\ 2\pi]. \end{aligned} \tag{20}$$

Taking in (20) $x = \frac{\pi}{2}$ and noticing that

$$\cos \frac{n\pi}{2} = \begin{cases} 0, & n = 2m-1; \\ (-1)^m, & n = 2m, \end{cases}$$

we conclude that

$$D(2k) = \frac{(-1)^{k+1}(4k-1)\pi^{2k}}{2 \cdot (2k)!} + \sum_{j=1}^{k} \frac{(-1)^{k+j+1} 4^j \pi^{2k-2j}}{(2k-2j)!} \zeta(2j). \tag{21}$$

Taking $x = \pi$ in (20) we conclude that

$$D(2k) = \frac{(-1)^{k+1}\pi^{2k}}{4k \cdot (2k-2)!} + \sum_{j=1}^{k} \frac{(-1)^{k+j+1} \pi^{2k-2j}}{(2k-2j)!} \zeta(2j). \tag{22}$$

Taking $x = \frac{3\pi}{2}$ in (20) and noticing that

$$(-1)^k \sum_{n=1}^{\infty} \frac{\cos \frac{3n\pi}{2}}{n^{2k}} = (-1)^{k+1} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n)^{2k}} = \frac{(-1)^{k+1}}{2^{2k}} D(2k),$$

we conclude that

$$D(2k) = \frac{(-1)^{k+1}(4k-3)3^{2k-1}\pi^{2k}}{2 \cdot (2k)!} + \sum_{j=1}^{k} \frac{(-1)^{k+j+1} 4^j (3\pi)^{2k-2j}}{(2k-2j)!} \zeta(2j). \tag{23}$$

Integrating on each side of (20) from 0 to 2π, we get

$$\zeta(2k) = \frac{(-1)^{k+1}(2k-1)2^{2k-2}\pi^{2k}}{(2k+1)!} + \sum_{j=1}^{k-1} \frac{(-1)^{k+j+1}(2\pi)^{2k-2j}}{(2k-2j+1)!} \zeta(2j). \tag{24}$$

Integrating term-by-term on each side of (20) from 0 to x, we get

$$\begin{aligned} &-\frac{x^{2k+1}}{2\cdot(2k+1)!}+\frac{\pi x^{2k}}{2\cdot(2k)!}+\sum_{j=1}^{k}\frac{(-1)^j x^{2k-2j+1}}{(2k-2j+1)!}\zeta(2j)\\ &=(-1)^k\sum_{n=1}^{\infty}\frac{\sin nx}{n^{2k+1}}, x\in[0,2\pi]. \end{aligned} \tag{25}$$

Taking $x=\frac{\pi}{2}$ a in (25) nd noticing that

$$\sin\frac{n\pi}{2}=\begin{cases}(-1)^{m-1}, & n=2m-1;\\ 0, & n=2m,\end{cases}$$

we conclude that

$$\begin{aligned} \sigma(2k+1)=&\frac{(-1)^k(4k+1)}{2\cdot(2k+1)!}\left(\frac{\pi}{2}\right)^{2k+1}\\ &+\sum_{j=1}^{k}\frac{(-1)^{k+j}}{(2k-2j+1)!}\left(\frac{\pi}{2}\right)^{2k-2j+1}\zeta(2j). \end{aligned} \tag{26}$$

Taking $x=\pi$ in (25) we conclude that

$$\zeta(2k)=\frac{(-1)^{k+1}k\pi^{2k}}{(2k+1)!}+\sum_{j=1}^{k-1}\frac{(-1)^{k+j+1}\pi^{2k-2j}}{(2k-2j+1)!}\zeta(2j). \tag{27}$$

Taking $x=\frac{3\pi}{2}$ in (25) and noticing that

$$(-1)^k\sum_{n=1}^{\infty}\frac{\sin\frac{3n\pi}{2}}{n^{2k+1}}=(-1)^{k+1}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^{2k+1}}=(-1)^{k+1}\sigma(2k+1),$$

we conclude that

$$\begin{aligned} \sigma(2k+1)=&\frac{(-1)^{k+1}(4k-1)\pi^{2k+1}}{4\cdot(2k+1)!}\left(\frac{3}{2}\right)^{2k}\\ &+\sum_{j=1}^{k}\frac{(-1)^{k+j+1}}{(2k-2j+1)!}\left(\frac{3\pi}{2}\right)^{2k-2j+1}\zeta(2j). \end{aligned} \tag{28}$$

Integrating on each side of (25) from 0 to 2π, we get

$$\zeta(2k)=\frac{(-1)^{k+1}k(2\pi)^{2k}}{(2k+2)!}+\sum_{j=1}^{k-1}\frac{(-1)^{k+j+1}2^{2k-2j+1}\pi^{2k-2j}}{(2k-2j+2)!}\zeta(2j). \tag{29}$$

The proof of Theorem 1 is complete.

Remark Form recurrence formula (14), or (16), or (18) we can obtain values of $\zeta(2k)$, for examples,

$$\zeta(2)=\frac{\pi^2}{6},\ \zeta(4)=\frac{\pi^4}{90},\ \zeta(6)=\frac{\pi^6}{945},\ \zeta(8)=\frac{\pi^8}{9450},\ \zeta(10)=\frac{\pi^{10}}{93555}.$$

By using values of $\zeta(2k)$, we can conclude values of $D(2k)$ from (11), or (12), or (13), for examples,

$$D(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12},$$
$$D(4) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^4} = \frac{7\pi^4}{720},$$
$$D(6) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^6} = \frac{31\pi^6}{30240}.$$

By using values of $\zeta(2k)$, we can also conclude values of $\sigma(2k)$ from (15), or (17), for examples,

$$\sigma(1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4},$$
$$\sigma(3) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3} = \frac{\pi^3}{32},$$
$$\sigma(5) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^5} = \frac{5\pi^5}{1536}.$$

Theorem 2. For $k \in \mathbb{N}$, we have

$$D(2k) = \frac{1}{4^k - 1} \left[\frac{(-1)^{k+1}\pi^{2k}}{2 \cdot (2k)!} + + \sum_{j=1}^{k-1} \frac{(-1)^{k+j+1} 4^j \pi^{2k-2j}}{(2k-2j)!} D(2j) \right], \tag{30}$$

$$D(2k) = \frac{(-1)^{k+1}\pi^{2k}}{2 \cdot (2k+1)!} + \sum_{j=1}^{k-1} \frac{(-1)^{k+j+1}\pi^{2k-2j}}{(2k-2j+1)!} D(2j), \tag{31}$$

$$\sigma(2k+1) = \frac{(-1)^k}{2 \cdot (2k+1)!} \left(\frac{\pi}{2}\right)^{2k+1} + \sum_{j=1}^{k} \frac{(-1)^{k+j}}{(2k-2j+1)!} \left(\frac{\pi}{2}\right)^{2k-2j+1} D(2j), \tag{32}$$

$$\zeta(2k) = \frac{(-1)^{k+1}\pi^{2k}}{2 \cdot (2k)!} + \sum_{j=1}^{k} \frac{(-1)^{k+j+1}\pi^{2k-2j}}{(2k-2j)!} D(2j). \tag{33}$$

Proof. Define the function g by

$$g(x) = x, \quad x \in (-\pi, \pi).$$

Easy computation reveals the Fourier series of f on $(-\pi, \pi)$:

$$x = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin nx}{n}, \quad x \in (-\pi, \pi). \tag{34}$$

Integration term-by-term yields

$$\frac{x^2}{2 \cdot 2!} - D(2) = -\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cos nx}{n^2}, \quad x \in [-\pi, \pi].$$

Clearly, if we integrate $2k-1$ times on each side of (34) from 0 to x, then we obtain

$$\frac{x^{2k}}{2\cdot(2k)!}+\sum_{j=1}^{k}\frac{(-1)^j x^{2k-2j}}{(2k-2j)!}D(2j)=(-1)^k\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos nx}{n^{2k}},\ x\in[-\pi,\ \pi]. \tag{35}$$

Taking $x=\pi/2$ in (35) we conclude that

$$D(2k)=\frac{1}{4^k-1}\left[\frac{(-1)^{k+1}\pi^{2k}}{2\cdot(2k)!}++\sum_{j=1}^{k-1}\frac{(-1)^{k+j+1}4^j\pi^{2k-2j}}{(2k-2j)!}D(2j)\right]. \tag{36}$$

Integrating on each side of (35) from 0 to π, we conclude that

$$D(2k)=\frac{(-1)^{k+1}\pi^{2k}}{2\cdot(2k+1)!}+\sum_{j=1}^{k-1}\frac{(-1)^{k+j+1}\pi^{2k-2j}}{(2k-2j+1)!}D(2j). \tag{37}$$

Integrating term-by-term on each side of (35) from 0 to x, we get

$$\begin{aligned}&\frac{x^{2k+1}}{2\cdot(2k+1)!}+\sum_{j=1}^{k}\frac{(-1)^j x^{2k-2j+1}}{(2k-2j+1)!}D(2j)\\&=(-1)^k\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin nx}{n^{2k+1}},x\in[-\pi,\pi].\end{aligned} \tag{38}$$

Taking $x=\pi/2$ in (38) we conclude that

$$\sigma(2k+1)=\frac{(-1)^k}{2\cdot(2k+1)!}\left(\frac{\pi}{2}\right)^{2k+1}+\sum_{j=1}^{k}\frac{(-1)^{k+j}}{(2k-2j+1)!}\left(\frac{\pi}{2}\right)^{2k-2j+1}D(2j). \tag{39}$$

Integrating on each side of (38) from 0 to π, we get

$$\zeta(2k)=\frac{(-1)^{k+1}\pi^{2k}}{2\cdot(2k)!}+\sum_{j=1}^{k}\frac{(-1)^{k+j+1}\pi^{2k-2j}}{(2k-2j)!}D(2j). \tag{40}$$

The proof of Theorem 2 is complete.

Theorem 3. For $k\in\mathbb{N}$, we have

$$\sigma(2k+1)=\frac{(-1)^k}{2\cdot(2k)!}\left(\frac{\pi}{2}\right)^{2k+1}+\sum_{j=1}^{k}\frac{(-1)^{k+j}}{(2k-2j+1)!}\left(\frac{\pi}{2}\right)^{2k-2j+1}\delta(2j), \tag{41}$$

$$\delta(2k)=\frac{(-1)^{k+1}\pi^{2k}}{8\cdot(2k-1)!}+\sum_{j=1}^{k-1}\frac{(-1)^{k+j+1}\pi^{2k-2j}}{2\cdot(2k-2j)!}\delta(2j), \tag{42}$$

$$\delta(2k)=\frac{(-1)^k}{2\cdot(2k-1)!}\left(\frac{\pi}{2}\right)^{2k}+\sum_{j=1}^{k-1}\frac{(-1)^{k+j}}{(2k-2j)!}\left(\frac{\pi}{2}\right)^{2k-2j}\delta(2j), \tag{43}$$

$$\delta(2k)=\frac{(-1)^{k+1}\pi^{2k}}{4\cdot(2k)!}+\sum_{j=1}^{k-1}\frac{(-1)^{k+j+1}\pi^{2k-2j}}{(2k-2j+1)!}\delta(2j). \tag{44}$$

Proof. Define the function h by

$$h(x) = |x|, \quad x \in [-\pi, \pi].$$

Easy computation reveals the Fourier series of f on $[-\pi, \pi]$:

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}, \ x \in [-\pi, \pi]. \tag{45}$$

Taking in (45) $x = 0$, we get

$$\delta(2) = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

Thus we have from (45)

$$\frac{\pi}{4}x - \delta(2) = -\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}, \ x \in [0, \pi]. \tag{46}$$

Integrating term-by-term, we have from (46)

$$\frac{\pi}{4} \cdot \frac{x^2}{2!} - \delta(2)x = -\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^3}, \ x \in [0, \pi].$$

Clearly, if we integrate $2k-1$ times on each side of (46) from 0 to x, then we obtain

$$\frac{\pi}{4} \cdot \frac{x^{2k}}{(2k)!} + \sum_{j=1}^{k} \frac{(-1)^j x^{2k-2j+1}}{(2k-2j+1)!} \delta(2j) = (-1)^k \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^{2k+1}}, \ x \in [0, \pi]. \tag{47}$$

Taking in (47) $x = \pi/2$ we conclude that

$$\sigma(2k+1) = \frac{(-1)^k}{2 \cdot (2k)!} \left(\frac{\pi}{2}\right)^{2k+1} + \sum_{j=1}^{k} \frac{(-1)^{k+j}}{(2k-2j+1)!} \left(\frac{\pi}{2}\right)^{2k-2j+1} \delta(2j). \tag{48}$$

Integrating on each side of (47) from 0 to π, we conclude that

$$\delta(2k) = \frac{(-1)^{k+1}\pi^{2k}}{8 \cdot (2k-1)!} + \sum_{j=1}^{k-1} \frac{(-1)^{k+j+1}\pi^{2k-2j}}{2 \cdot (2k-2j)!} \delta(2j). \tag{49}$$

Integrating term-by-term on each side of (47) from 0 to x, we obtain that

$$\begin{aligned} &\frac{\pi}{4} \cdot \frac{x^{2k+1}}{(2k+1)!} + \sum_{j=1}^{k+1} \frac{(-1)^j x^{2k-2j+2}}{(2k-2j+2)!} \delta(2j) \\ &= (-1)^{k+1} \sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^{2k+2}}, \ x \in [0, \pi]. \end{aligned} \tag{50}$$

Taking $x = \pi/2$ in (50), we conclude that

$$\delta(2k) = \frac{(-1)^k}{2 \cdot (2k-1)!} \left(\frac{\pi}{2}\right)^{2k} + \sum_{j=1}^{k-1} \frac{(-1)^{k+j}}{(2k-2j)!} \left(\frac{\pi}{2}\right)^{2k-2j} \delta(2j). \tag{51}$$

Integrating on each side of (50) from 0 to π, we conclude that

$$\delta(2k) = \frac{(-1)^{k+1}\pi^{2k}}{4\cdot(2k)!} + \sum_{j=1}^{k-1} \frac{(-1)^{k+j+1}\pi^{2k-2j}}{(2k-2j+1)!}\delta(2j). \tag{52}$$

The proof of Theorem 3 is complete.

Remark. Form recurrence formula (42), (43) or (44), we can obtain values of $\delta(2k)$, for examples,

$$\delta(2) = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8},$$
$$\delta(4) = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^4}{96},$$
$$\delta(6) = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} = \frac{\pi^6}{960}.$$

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Smarandache Semiquasi Near-rings[1]

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Abstract G. Pilz [1] has defined near-rings and semi-near-rings. In this paper we introduce the concepts of quasi-near ring and semiquasi-near ring. We have also defined Smarandache semiquasi-near-ring. Some examples are constructed. We have posed some open problems.

Keywords Near-ring, semi-near-ring, quasi-near-ring, semiquasi-near-ring, Smarandache semiquasi-near-ring.

§1. Introduction

In the paper [2] W.B. Kandasamy has introduced a new concept of Smarandache semi-near ring. These are associative rings. We have defined a new concepts of quasi-near ring and Smarandache semiquasi-near-ring. These are non associative rings.

Definition 1.1. An algebraic structure $(Q, +, \cdot)$ is called a quasi-near-ring (or a right quasi-near-ring) if it satisfies the following three conditions:

1.$(Q, +)$ is a group (not necessarily abelian).

2.$(Q, \cdot)$ is a quasigroup.

3.$(n_1 + n_2) \cdot n_3 = n_1 \cdot n_3 + n_2 \cdot n_3$ for all $n_1, n_2, n_3 \in Q$ (right distributive law).

Example 1.1. Let $Q = \{1, 2, 3, 4\}$ and the two binary operations are defined on Q by the following tables;

+	1	2	3	4
1	1	2	3	4
2	2	3	4	1
3	3	4	1	2
4	4	1	2	3

.	1	2	3	4
1	4	2	1	3
2	1	3	4	2
3	3	1	2	4
4	2	4	3	1

Definition 1.2. An algebraic system $(S, +, \cdot)$ is called a semiquasi-near-ring (or right semiquasi-near-ring) if it satisfies the following three conditions:

1. $(S, +)$ is a quasigroup (not necessarily abelian).

2. $(S, \cdot)$ is a quasigroup.

3. $(n_1 + n_2) \cdot n_3 = n_1 \cdot n_3 + n_2 \cdot n_3$ for all $n_1, n_2, n_3 \in S$ (right distributive law).

Example 1.2. Consider the algebraic system $(S, +, \cdot)$ where $S = \{1, 2, 3, 4\}$ defined by the following tables;

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+	1	2	3	4
1	1	3	4	2
2	4	2	1	3
3	2	4	3	1
4	3	1	2	4

.	1	2	3	4
1	4	2	1	3
2	1	3	4	2
3	3	1	2	4
4	2	4	3	1

Example 1.3. We know that integers **Z** with subtraction (-) forms a quasigroup. $(\mathbf{Z}, \cdot)$ is a quasigroup and subtraction of integers distributes over multiplication. Thus $(\mathbf{Z}, -, \cdot)$ is a semiquasi-near-ring.

Definition 1.3 We know that integers **Z** with subtraction (-) forms a quasigroup. $(\mathbf{Z}, \cdot)$ is a quasigroup and subtraction of integers distributes over multiplication. Thus $(\mathbf{Z}, -, \cdot)$ is a semiquasi-near-ring.

Example 1.4. Consider the semiquasi-near-ring $(S, +, \cdot)$ defined by the following tables;

+	1	2	3	4
1	1	3	4	2
2	4	2	1	3
3	2	4	3	1
4	3	1	2	4

.	1	2	3	4
1	4	3	1	2
2	1	2	4	3
3	3	4	2	1
4	2	1	3	4

one can easily verify that addition distributes over multiplication from right as well as S contains $N = \{4\}$ properly which is a quasi-near-ring.
Thus $(S, +, \cdot)$ is a Smarandache semiquasi-near-ring.

Example 1.5. Let R be the set of reals. We know that $(R, +)$ is a group and hence a quasigroup. Also, R w.r.t. division is a quasigroup, that is $(R, \div)$ is a quasigroup. More over, addition distributes over division from right. Thus $(R, +, \div)$ is a semiquasi-near-ring.

Let Q be the set of non-zero rationals. Then $(Q, +)$ is a group. Also, $(Q, \div)$ is a quasigroup. Addition distributes over division. Hence $(Q, +, \div)$ is a quasi-near-ring.
We know that $R \supset Q$. Therefore, $(R, +, \div)$ is a Smarandache semiquasi-near-ring. We now show by an example that there do exist semiquasi-near -rings which are not Smarandache semiquasi-near-rings.
Consider example 1.2 where we can not have a quasi-near-ring contained in S.
We give below the example of a smallest Smarandache semiquasi-near-ring which is not a near-ring.

Example 1.6. Consider the semiquasi-near ring $(S = \{1, 2, 3\}, +, \cdot)$ defined by the following tables;

+	1	2	3
1	1	3	2
2	3	2	1
3	2	1	3

.	1	2	3
1	3	2	1
2	1	3	2
3	2	1	3

One can easily verify that $(S = \{1, 2, 3\}, +, \cdot)$ is a semiquasi-near-ring. Moreover, $N = \{3\} \subset S$ and $(N, +, \cdot)$ is a quasi-near ring. Therefore $(S = \{1, 2, 3\}, +, \cdot)$ is a Smarandache semiquasi-near-ring.

We now show by an example that there do exist semiquasi-near -rings which are not Smarandache semiquasi-near-rings.

Consider example 1.2 where we can not have a quasi-near-ring contained in S.

We give below the example of a smallest Smarandache semiquasi-near-ring which is not a near-ring.

Definition 1.4. N is said to be an Anti-Smarandache semiquasi-near-ring if N is a quasi-near-ring and has a proper subset A such that A is a semiquasi-near-ring under the same operations as of N.

Example 1.7. In example 1.5 $(R, +, \div)$ is also a quasi-near-ring which contains a semiquasi-near-ring $(Q, +, \div)$.

Thus we can say that $(R, +, \div)$ is an Anti-Smarandache semiquasi-near-ring.

We propose the following:

Problem 1. Do there exist a finite Smarandache semiquasi-near-ring such that the order of the quasi-near-ring contained in it is greater than 1 ?

Problem 2. How to construct finite Anti-Smarandache semiquasi-near-rings ?

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On exponentially harmonic numbers

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Abstract In this note there are introduced two new concepts of e-harmonic numbers, and a new concept of e-perfect number. Their initial study is provided.

Keywords Exponential divisors, harmonic numbers, e-perfect numbers.

§1. Introduction

Let $\sigma(n)$ and $d(n)$ denote the sum, resp. number of divisors of n. In 1948 O. Ore [12] called a number n **harmonic** if

$$\sigma(n)|nd(n) \tag{1}$$

See e.g. G. L. Cohen and R. M. Sorli [2] for such numbers. If $\sigma_k(n)$ denotes the sum of k-th powers of divisors of n ($k \geq 1$ integer), then G. L. Cohen and D. Moujie [3] introduced **k-harmonic** numbers by

$$\sigma_k(n)|nd(n) \tag{2}$$

A perfect number is always harmonic (see [12]), and Ore conjectured that all harmonic numbers are even. This is a quite deep conjecture, since, if true, clearly would imply the non-existence of odd perfect numbers.

A divisor d of n is called **unitary divisor** if $\left(d, \frac{n}{d}\right) = 1$. If $\sigma^*(n)$, $d^*(n)$ are the sum, resp. number of unitary divisors of n, then n is called **unitary harmonic** if

$$\sigma^*(n)|nd^*(n), \tag{3}$$

see K. Nageswara Rao [11], P. Hagis and G. Lord [5], Ch. Wall [19].

A divisor d of n is called a bi-unitary divisor, if the greatest common unitary divisor of d and $\frac{n}{d}$ is 1. If $\sigma^{**}(n)$, $d^{**}(n)$ are the sum, and number of bi-unitary divisors of n, recently we have introduced (see [16]) **bi-unitary harmonic** numbers by

$$\sigma^{**}(n)|nd^{**}(n) \tag{4}$$

For **infinitary harmonic** numbers, related to the concept of an infinitary divisor, see P. Hagis and G. L. Cohen [7].

Let $n > 1$ be a positive integer having the prime factorization $n = p_1^{a_1} \dots p_r^{a_r}$. A divisor d of n is called exponential divisor, if $d = p_1^{b_1} \dots p_r^{b_r}$ where $b_1|a_1, \dots, b_r|a_r$. This notion is due to E. G. Straus and M. V. Subbarao [17]. Let $\sigma_e(n)$ and $d_e(n)$ be the sum and number of,

exponential divisors of n. Let by convention $\sigma_e(1) = d_e(1) = 1$. Straus and Subbarao have introduced e**-perfect** numbers n by

$$\sigma_e(n) = 2n. \tag{5}$$

They proved the non-existence of odd e-perfect numbers, with related other results. For results on e-superperfect numbers (i.e. satisfying $\sigma_e(\sigma_e(n)) = 2n$), see [8]. For density problems, e-perfect numbers not divisible by 3, or e-multiperfect numbers, see P. Hagis [6], L. Lucht [10], J. Fabrykowski and M. V. Subbarao [4], W. Aiello et al. [1]. For results on $d_e(n)$, we quote J. M. DeKoninck and A. Ivić [9]. For the exponential totient function $\varphi_e(n)$, see J. Sándor [13]. For e-convolution and a survey connected to the Möbius function, see J. Sándor and A. Bege [14]. For multiplicatively e-perfect numbers, see J. Sándor [15].

§2. Exponential harmonic numbers

The aim of this note is study two notions of e**-harmonic numbers**. An integer n will be called e**-harmonic of type 1** if

$$\sigma_e(n) | n d_e(n) \tag{6}$$

In all examples of section 1 the harmonic numbers notions were suggested by the consideration of the **harmonic means** of the considered divisors. For example, if $1 = d_1 < d_2 < \cdots < d_r = n$ are all divisors of n, then their harmonic mean is

$$H(n) = r / \left(\frac{1}{d_1} + \cdots + \frac{1}{d_r} \right).$$

Since

$$\sum \frac{1}{d_r} = \frac{1}{n} \sum \frac{n}{d_r} = \frac{1}{n} \sum d_r = \frac{\sigma(n)}{n},$$

we get

$$H(n) = \frac{n d(n)}{\sigma(n)}, \tag{7}$$

so a harmonic number n is a number such that $H(n)$ is an integer. E.g. for $H(n) = 2$ we get the so-called **"balanced numbers"** proposed by M. V. Subbarao [18], with single solution $n = 6$.

Now, if we consider the harmonic mean $H_e(n)$ of the exponential divisors $d_1^{(e)}, \ldots, d_r^{(e)}$, then

$$H_e(n) = \frac{r}{\sum \frac{1}{d_i^{(e)}}},$$

where $r = d_e(n)$ denotes the number of exponential (or e-) divisors of n. Let p^a be a prime power. Then the e-divisors of p^a are p^d with $d|a$, so

$$\sum \frac{1}{d_i^{(e)}} = \sum_{d|a} \frac{1}{p^a} = \frac{1}{p^a} \sum_{d|a} p^{a-d}$$

in this case. When $n = p^a q^b$ ($p \neq q$ primes) one obtains similarly

$$\sum \frac{1}{d_i^{(e)}} = \sum_{d_1|a, d_2|b} \frac{1}{p^{d_1} q^{d_2}} = \left(\sum_{d_1|a} \frac{1}{p^{d_1}} \right) \left(\sum_{d_2|b} \frac{1}{q^{d_2}} \right)$$

$$= \frac{1}{p^a q^b}\left(\sum_{d_1|a} p^{a-d_1}\right)\left(\sum_{d_2|b} q^{b-d_2}\right).$$

In the general case, when $n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$, one has

$$H_e(n) = \frac{n d_e(n)}{S_e(n)}, \tag{8}$$

where

$$S_e(n) = \prod_{i=1}^{r}\left(\sum_{d_i|a_i} p_i^{a_i - d_i}\right) \tag{9}$$

We say that n is **e-harmonic of type 2** if $H_e(n)$ is integer, i.e.

$$S_e(n) | n d_e(n), \tag{10}$$

where $S_e(n)$ is given by (9).

Theorem 1. If n is squarefree, then it is e-harmonic of both types.

Proof. If n is squarefree, i.e. $n = p_1 p_2 \dots p_r$, then clearly by definitions of $\sigma_e(n)$ and $S_e(n)$ one has $\sigma_e(n) = p_1 p_2 \dots p_r = n$ and $S_e(n) = \prod_{i=1}^{r}\left(\sum_{d_i|1} p_i^{1-d_i}\right) = 1$, so (6) and (8) are satisfied. We shall see later (see the Remark after Theorem 3), that there exist also numbers with this property, which are not squarefree.

Theorem 2. Let $n = p_1^{a_1} \dots p_r^{a_r}$ be the prime factorization of $n > 1$. If n is e-perfect, then n is e-harmonic of type 1 if and only if at least one of $a_1, \dots, a_r$ is not a perfect square.

Proof. If $\sigma_e(n) = 2n$, then (6) gives $2|d_e(n)$. It is well-known that $d_e(n) = d(a_1) \dots d(a_r)$, so at least one of $d(a_1), \dots, d(a_r)$ must be even. But, it is well-known that $d(a)$ is even iff a is not a perfect square, so the result follows.

Remark. 1) Since e.g. $2^2 \cdot 3^3 \cdot 5^2$, $2^3 \cdot 3^2 \cdot 5^2$, $2^4 \cdot 3^3 \cdot 5^2 \cdot 11^2$, $2^6 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 13^2$, $2^7 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 13^2$ are e-perfect numbers, by the above theorem, these are also e-harmonic numbers of type 1.

2) Similarly, if n is $e-k$ perfect, i.e. $\sigma_e(n) = kn$ ($k \geq 2$ integer), see [**?**], then n is e-harmonic of type 1 iff

$$k | d(a_1) d(a_2) \dots d(a_r) \tag{11}$$

In what follows we shall introduce another new notion. We say that n is **modified e-perfect** number, if

$$S_e(n) | n, \tag{12}$$

where $S_e(n)$ is given by (9). Since $S_e(p) = 1$, $S_e(p^2) = p + 1$, $S_e(p^3) = p^2 + 1$, we have $S_e(2^2 \cdot 3^2) = (2+1)(3+1) = 2^2 \cdot 3$, $S_e(2^2 \cdot 3^3 \cdot 5^2) = (2+1)(3^2+1)(5+1) = 2^2 \cdot 3^2 \cdot 5$, $S_e(2^3 \cdot 3^2 \cdot 5^2) = (2^2+1)(3+1)(5+1) = 2^3 \cdot 3 \cdot 5$, so $2^2 \cdot 3^2$, $2^2 \cdot 3^3 \cdot 5^2$, $2^3 \cdot 3^2 \cdot 5^2$ (which are also e-perfect) are modified e-perfect numbers. Clearly $n = 2^4 \cdot 3^2 \cdot 11^2$ (which is e-perfect) is not modified e-perfect, since by $S_e(p^4) = 1 + p^2 + p^3$, we have $S_e(2^4) = 1 + 2^2 + 2^3 = 13$.

Theorem 3. If n is modified e-perfect, then it is harmonic of type 2.

Proof. This follows at once from (12) and (8).

Remark. Thus $2^2 \cdot 3^3 \cdot 5^2$ and $2^3 \cdot 3^2 \cdot 5^2$ are e-harmonic numbers of both types (though they are not squarefree).

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Parastrophic invariance of Smarandache quasigroups

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Abstract Every quasigroup $(L, \cdot)$ belongs to a set of 6 quasigroups, called parastrophes denoted by (L, π_i), $i \in \{1, 2, 3, 4, 5, 6\}$. It is shown that (L, π_i) is a Smarandache quasigroup with associative subquasigroup (S, π_i) $\forall$ $i \in \{1, 2, 3, 4, 5, 6\}$ if and only if for any of some four $j \in \{1, 2, 3, 4, 5, 6\}$, (S, π_j) is an isotope of (S, π_i) or (S, π_k) for one $k \in \{1, 2, 3, 4, 5, 6\}$ such that $i \neq j \neq k$. Hence, (L, π_i) is a Smarandache quasigroup with associative subquasigroup (S, π_i) $\forall$ $i \in \{1, 2, 3, 4, 5, 6\}$ if and only if any of the six Khalil conditions is true for any of some four of (S, π_i).

Keywords Parastrophes, Smarandache quasigroups, isotopic.

§1. Introduction

The study of the Smarandache concept in groupoids was initiated by W.B. Vasantha Kandasamy in [18]. In her book [16] and first paper [17] on Smarandache concept in loops, she defined a Smarandache loop as a loop with at least a subloop which forms a subgroup under the binary operation of the loop. Here, the study of Smarandache quasigroups is continued after their introduction in Muktibodh [9] and [10]. Let L be a non-empty set. Define a binary operation $(\cdot)$ on L : if $x \cdot y \in L$ $\forall$ $x, y \in L$, $(L, \cdot)$ is called a groupoid. If the system of equations ; $a \cdot x = b$ and $y \cdot a = b$ have unique solutions for x and y respectively, then $(L, \cdot)$ is called a quasigroup. Furthermore, if $\exists$ a ! element $e \in L$ called the identity element such that $\forall$ $x \in L$, $x \cdot e = e \cdot x = x$, $(L, \cdot)$ is called a loop. It can thus be seen clearly that quasigroups lie in between groupoids and loops. So, the Smarandache concept needed to be introduced into them and studied since it has been introduced and studied in groupoids and loops. Definitely, results of the Smarandache concept in groupoids will be true in quasigroup that are Smarandache and these together will be true in Smarandache loops.

It has been noted that every quasigroup $(L, \cdot)$ belongs to a set of 6 quasigroups, called adjugates by Fisher and Yates [6], conjugates by Stein [15], [14] and Belousov [2] and parastrophes by Sade [12]. They have been studied by Artzy [1] and a detailed study on them can be found in [11], [4] and [5]. So for a quasigroup $(L, \cdot)$, its parastrophes are denoted by (L, π_i), $i \in \{1, 2, 3, 4, 5, 6\}$ hence one can take $(L, \cdot) = (L, \pi_1)$. For more on quasigroup, loops and their properties, readers should check [11], [3], [4], [5], [7] and [16]. Let $(G, \oplus)$ and $(H, \otimes)$ be two distinct quasigroups. The triple (A, B, C) such that A, B, C : $(G, \oplus) \to (H, \otimes)$ are bijections

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is said to be an isotopism if and only if $xA \otimes yB = (x \oplus y)C \ \forall \ x, y \in G$. Thus, H is called an isotope of G and they are said to be isotopic.

In this paper, it will be shown that (L, π_i) is a Smarandache quasigroup with associative subquasigroup $(S, \pi_i) \ \forall \ i \in \{1, 2, 3, 4, 5, 6\}$ if and only if for any of some four $j \in \{1, 2, 3, 4, 5, 6\}$, (S, π_j) is an isotope of (S, π_i) or (S, π_k) for one $k \in \{1, 2, 3, 4, 5, 6\}$ such that $i \neq j \neq k$. Hence, it can be concluded that (L, π_i) is a Smarandache quasigroup with associative subquasigroup $(S, \pi_i) \ \forall \ i \in \{1, 2, 3, 4, 5, 6\}$ if and only if any of the six Khalil conditions is true for any of some four of (S, π_i).

§2. Preliminaries

Definition 2.1 Let $(L, \cdot)$ be a quasigroup. If there exists at least a non-trivial subset $S \subset L$ such that $(S, \cdot)$ is an associative subquasigroup in L, then L is called a Smarandache quasigroup (SQ).

Definition 2.2 Let (L, θ) be a quasigroup. The 5 parastrophes or conjugates or adjugates of (L, θ) are quasigroups whose binary operations θ^* , θ^{-1} , ${}^{-1}\theta$, $(\theta^{-1})^*$, $({}^{-1}\theta)^*$ defined on L are given by :

(a) $(L, \theta^*) \ : \ y\theta^* x = z \Leftrightarrow x\theta y = z \ \forall \ x, y, z \in L.$

(b) $(L, \theta^{-1}) \ : \ x\theta^{-1} z = y \Leftrightarrow x\theta y = z \ \forall \ x, y, z \in L.$

(c) $(L, {}^{-1}\theta) \ : \ z \ {}^{-1}\theta y = x \Leftrightarrow x\theta y = z \ \forall \ x, y, z \in L.$

(d) $(L, (\theta^{-1})^*) \ : \ z(\theta^{-1})^* x = y \Leftrightarrow x\theta y = z \ \forall \ x, y, z \in L.$

(e) $(L, ({}^{-1}\theta)^*) \ : \ y({}^{-1}\theta)^* z = x \Leftrightarrow x\theta y = z \ \forall \ x, y, z \in L.$

Definition 2.3 Let (L, θ) be a loop.

(a) R_x and L_x represent the left and right translation maps in $(L, \theta) \ \forall \ x \in L$.

(b) R_x^* and L_x^* represent the left and right translation maps in $(L, \theta^*) \ \forall \ x \in L$.

(c) $\mathcal{R}_x$ and $\mathcal{L}_x$ represent the left and right translation maps in $(L, \theta^{-1}) \ \forall \ x \in L$.

(d) $\mathbb{R}_x$ and $\mathbb{L}$ represent the left and right translation maps in $(L, {}^{-1}\theta) \ \forall \ x \in L$.

(e) $\mathcal{R}_x^*$ and $\mathcal{L}_x^*$ represent the left and right translation maps in $(L, (\theta^{-1})^*) \ \forall \ x \in L$.

(f) $\mathbb{R}_x^*$ and $\mathbb{L}^*$ represent the left and right translation maps in $(L, ({}^{-1}\theta)^*) \ \forall \ x \in L$.

Remark 2.1 If (L, θ) is a loop, (L, θ^*) is also a loop(and vice versa) while the other adjugates are quasigroups.

Lemma 2.1 If (L, θ) is a quasigroup, then

1. $R_x^* = L_x$, $L_x^* = R_x$, $\mathcal{L}_x = L_x^{-1}$, $\mathbb{R}_x = R_x^{-1}$, $\mathcal{R}_x^* = L_x^{-1}$, $\mathbb{L}_x^* = R_x^{-1} \ \forall \ x \in L$.

2. $\mathcal{L}_x = R_x^{*-1}$, $\mathbb{R}_x = L_x^{*-1}$, $\mathcal{R}_x^* = R_x^{*-1} = \mathcal{L}_x$, $\mathbb{L}_x^* = L_x^{*-1} = \mathbb{R}_x \ \forall \ x \in L$.

Proof. The proof of these follows by using Definition 2.2 and Definition 2.3.

(1) $y\theta^* x = z \Leftrightarrow x\theta y = z \Rightarrow y\theta^* x = x\theta y \Rightarrow yR_x^* = yL_x \Rightarrow R_x^* = L_x$. Also, $y\theta^* x = x\theta y \Rightarrow xL_y^* = xR_y \Rightarrow L_y^* = R_y$.

$x\theta^{-1} z = y \Leftrightarrow x\theta y = z \Rightarrow x\theta(x\theta^{-1} z) = z \Rightarrow x\theta z\mathcal{L}_x = z \Rightarrow z\mathcal{L}_x L_x = z \Rightarrow \mathcal{L}_x L_x = I$. Also, $x\theta^{-1}(x\theta y) = y \Rightarrow x\theta^{-1} yL_x = y \Rightarrow yL_x\mathcal{L}_x = y \Rightarrow L_x\mathcal{L}_x = I$. Hence, $\mathcal{L}_x = L_x^{-1} \ \forall \ x \in L$.

$z({}^{-1}\theta)y = x \Leftrightarrow x\theta y = z \Rightarrow (x\theta y)({}^{-1}\theta)y = x \Rightarrow xR_y({}^{-1}\theta)y = x \Rightarrow xR_y\mathbb{R}_y = x \Rightarrow R_y\mathbb{R}_y = I$. Also, $(z({}^{-1}\theta)y)\theta y = z \Rightarrow z\mathbb{R}_y\theta y = z \Rightarrow z\mathbb{R}_y R_y = z \Rightarrow \mathbb{R}_y R_y = I$. Thence, $\mathbb{R}_y = R_y^{-1} \ \forall \ x \in L$.

$z(\theta^{-1})^* x = y \Leftrightarrow x\theta y = z$, so, $x\theta(z(\theta^{-1})^* x) = z \Rightarrow x\theta z\mathcal{R}_x^* = z \Rightarrow z\mathcal{R}_x^* L_x = z \Rightarrow \mathcal{R}_x^* L_x = I$.

Also, $(x\theta y)(\theta^{-1})^*x = y \Rightarrow yL_x(\theta^{-1})^*x = y \Rightarrow yL_x\mathcal{R}_x^* = y \Rightarrow L_y\mathcal{R}_x^* = I$. Whence, $\mathcal{R}_x^* = L_x^{-1}$. $y({}^{-1}\theta)^*z = x \Leftrightarrow x\theta y = z$, so, $y({}^{-1}\theta)^*(x\theta y) = x \Rightarrow y({}^{-1}\theta)^*xR_y = x \Rightarrow xR_y\mathbb{L}_y^* = x \Rightarrow R_y\mathbb{L}_y^* = I$. Also, $(y({}^{-1}\theta)^*z)\theta y = z \Rightarrow z\mathbb{L}_y^*\theta y = z \Rightarrow z\mathbb{L}_y^*R_y = z \Rightarrow \mathbb{L}_y^*R_y = I$. Thus, $\mathbb{L}_y^* = R_y^{-1}$.

(2) These ones follow from (1).

Lemma 2.2 Every quasigroup which is a Smarandache quasigroup has at least a subgroup.

Proof. If a quasigroup $(L, \cdot)$ is a SQ, then there exists a subquasigroup $S \subset L$ such that $(S, \cdot)$ is associative. According [8], every quasigroup satisfying the associativity law has an identity hence it is a group. So, S is a subgroup of L.

Theorem 2.1 (Khalil Conditions [13]) A quasigroup is an isotope of a group if and only if any one of some six identities are true in the quasigroup.

§3. Main Results

Theorem 3.1 (L, θ) is a Smarandache quasigroup of with associative subquasigroup (S, θ) if and only if any of the following equivalent statements is true.

1. (S, θ) is isotopic to $(S, (\theta^{-1})^*)$.
2. (S, θ^*) is isotopic to (S, θ^{-1}).
3. $(S, \theta)$is isotopic to $(S, ({}^{-1}\theta)^*)$.
4. (S, θ^*) is isotopic to $(S, {}^{-1}\theta)$.

Proof. L is a SQ with associative subquasigroup S if and only if $s_1\theta(s_2\theta s_3) = (s_1\theta s_2)\theta s_3 \Leftrightarrow R_{s_2}R_{s_3} = R_{s_2\theta s_3} \Leftrightarrow L_{s_1\theta s_2} = L_{s_2}L_{s_1}\ \forall\ s_1, s_2, s_3 \in S$.

The proof of the equivalence of (1) and (2) is as follows. $L_{s_1\theta s_2} = L_{s_2}L_{s_1} \Leftrightarrow \mathcal{L}_{s_1\theta s_2}^{-1} = \mathcal{L}_{s_2}^{-1}\mathcal{L}_{s_1}^{-1} \Leftrightarrow \mathcal{L}_{s_1\theta s_2} = \mathcal{L}_{s_1}\mathcal{L}_{s_2} \Leftrightarrow (s_1\theta s_2)\theta^{-1}s_3 = s_2\theta^{-1}(s_1\theta^{-1}s_3) \Leftrightarrow (s_1\theta s_2)\mathcal{R}_{s_3} = s_2\theta^{-1}s_1\mathcal{R}_{s_3} = s_1\mathcal{R}_{s_3}(\theta^{-1})^*s_2 \Leftrightarrow (s_1\theta s_2)\mathcal{R}_{s_3} = s_1\mathcal{R}_{s_3}(\theta^{-1})^*s_2 \Leftrightarrow (s_2\theta^*s_1)\mathcal{R}_{s_3} = s_2\theta^{-1}s_1\mathcal{R}_{s_3} \Leftrightarrow (\mathcal{R}_{s_3}, I, \mathcal{R}_{s_3})$: $(S, \theta) \to (S, (\theta^{-1})^*) \Leftrightarrow (I, \mathcal{R}_{s_3}, \mathcal{R}_{s_3})$: $(S, \theta^*) \to (S, \theta^{-1}) \Leftrightarrow (S, \theta)$ is isotopic to $(S, (\theta^{-1})^*) \Leftrightarrow (S, \theta^*)$ is isotopic to (S, θ^{-1}).

The proof of the equivalence of (3) and (4) is as follows. $R_{s_2}R_{s_3} = R_{s_2\theta s_3} \Leftrightarrow \mathbb{R}_{s_2}^{-1}\mathbb{R}_{s_3}^{-1} = \mathbb{R}_{s_2\theta s_3}^{-1} \Leftrightarrow \mathbb{R}_{s_3}\mathbb{R}_{s_2} = \mathbb{R}_{s_2\theta s_3} \Leftrightarrow ({s_1}^{-1}\theta s_3)^{-1}\theta s_2 = {s_1}^{-1}\theta(s_2\theta s_3) \Leftrightarrow (s_2\theta s_3)\mathbb{L}_{s_1} = s_3{\mathbb{L}_{s_1}}^{-1}\theta s_2 = s_2({}^{-1}\theta)^*s_3\mathbb{L}_{s_1} \Leftrightarrow (s_2\theta s_3)\mathbb{L}_{s_1} = s_2({}^{-1}\theta)^*s_3\mathbb{L}_{s_1} \Leftrightarrow (s_3\theta^*s_2)\mathbb{L}_{s_1} = s_3{\mathbb{L}_{s_1}}^{-1}\theta s_2 \Leftrightarrow (I, \mathbb{L}_{s_1}, \mathbb{L}_{s_1})$: $(S, \theta) \to (S, ({}^{-1}\theta)^*) \Leftrightarrow (\mathbb{L}_{s_1}, I, \mathbb{L}_{s_1})$: $(S, \theta^*) \to (S, {}^{-1}\theta) \Leftrightarrow (S, \theta)$ is isotopic to $(S, ({}^{-1}\theta)^*) \Leftrightarrow (S, \theta^*)$ is isotopic to $(S, {}^{-1}\theta)$.

Remark 3.1 In the proof of Theorem 3.1, it can be observed that the isotopisms are triples of the forms (A, I, A) and (I, B, B). All weak associative identities such as the Bol, Moufang and extra identities have been found to be isotopic invariant in loops for any triple of the form (A, B, C) while the central identities have been found to be isotopic invariant only under triples of the forms (A, B, A) and (A, B, B). Since associativity obeys all the Bol-Moufang identities, the observation in the theorem agrees with the latter stated facts.

Corollary 3.1 (L, θ) is a Smarandache quasigroup with associative subquasigroup (S, θ) if and only if any of the six Khalil conditions is true for some four parastrophes of (S, θ).

Proof. Let (L, θ) be the quasigroup in consideration. By Lemma 2.2, (S, θ) is a group. Notice that $R_{s_2}R_{s_3} = R_{s_2\theta s_3} \Leftrightarrow L_{s_2\theta s_3}^* = L_{s_3}^*L_{s_2}^*$. Hence, (S, θ^*) is also a group. In Theorem 3.1, two of the parastrophes are isotopes of (S, θ) while the other two are isotopes of (S, θ^*).

Since the Khalil conditions are neccessary and sufficient conditions for a quasigroup to be an isotope of a group, then they must be necessarily and sufficiently true in the four quasigroup parastrophes of (S, θ).

Lemma 3.1 (L, θ^*) is a Smarandache quasigroup with associative subquasigroup (S, θ^*) if and only if any of the following equivalent statements is true.

1. (S, θ^*) is isotopic to $(S, {}^{-1}\theta)$.
2. (S, θ) is isotopic to $(S, ({}^{-1}\theta)^*)$.
3. (S, θ^*) is isotopic to (S, θ^{-1}).
4. (S, θ) is isotopic to $(S, (\theta^{-1})^*)$.

Proof. Replace (L, θ) with (L, θ^*) in Theorem 3.1.

Corollary 3.2 (L, θ^*) is a Smarandache quasigroup with associative subquasigroup (S, θ^*) if and only if any of the six Khalil conditions is true for some four parastrophes of (S, θ).

Proof. Replace (L, θ) with (L, θ^*) in Corollary 3.1.

Lemma 3.2 (L, θ^{-1}) is a Smarandache quasigroup with associative subquasigroup (S, θ^{-1}) if and only if any of the following equivalent statements is true.

1. (S, θ^{-1}) is isotopic to (S, θ^*) .
2. $(S, (\theta^{-1})^*)$ is isotopic to (S, θ).
3. (S, θ^{-1}) is isotopic to $(S, {}^{-1}\theta)$.
4. $(S, (\theta^{-1})^*)$ is isotopic to $(S, ({}^{-1}\theta)^*)$.

Proof. Replace (L, θ) with (L, θ^{-1}) in Theorem 3.1.

Corollary 3.3 (L, θ^{-1}) is a Smarandache quasigroup with associative subquasigroup (S, θ^{-1}) if and only if any of the six Khalil conditions is true for some four parastrophes of (S, θ).

Proof. Replace (L, θ) with (L, θ^{-1}) in Corollary 3.1.

Lemma 3.3 $(L, {}^{-1}\theta)$ is a Smarandache quasigroup with associative subquasigroup $(S, {}^{-1}\theta)$ if and only if any of the following equivalent statements is true.

1. $(S, {}^{-1}\theta)$ is isotopic to (S, θ^{-1}).
2. $(S, ({}^{-1}\theta)^*)$ is isotopic to $(S, (\theta^{-1})^*)$.
3. $(S, {}^{-1}\theta)$ is isotopic to (S, θ^*).
4. $(S, ({}^{-1}\theta)^*)$ is isotopic to (S, θ).

Proof. Replace (L, θ) with $(L, {}^{-1}\theta)$ in Theorem 3.1.

Corollary 3.4 $(L, {}^{-1}\theta)$ is a Smarandache quasigroup with associative subquasigroup $(S, {}^{-1}\theta)$ if and only if any of the six Khalil conditions is true for some four parastrophes of (S, θ).

Proof. Replace (L, θ) with $(L, {}^{-1}\theta)$ in Corollary 3.1.

Lemma 3.4 $(L, (\theta^{-1})^*)$ is a Smarandache quasigroup with associative subquasigroup $(S, (\theta^{-1})^*)$ if and only if any of the following equivalent statements is true.

1. $(S, (\theta^{-1})^*)$ is isotopic to $(S, ({}^{-1}\theta)^*)$.
2. (S, θ^{-1}) is isotopic to $(S, {}^{-1}\theta)$.
3. $(S, (\theta^{-1})^*)$ is isotopic to (S, θ).
4. $(S, \theta^{-1}))$ is isotopic to (S, θ^*).

Proof. Replace (L, θ) with $(L, (\theta^{-1})^*)$ in Theorem 3.1.

Corollary 3.5 $(L, (\theta^{-1})^*)$ is a Smarandache quasigroup with associative subquasigroup $(S, (\theta^{-1})^*)$ if and only if any of the six Khalil conditions is true for some four parastrophes of (S, θ).

Proof. Replace (L, θ) with $(L, (\theta^{-1})^*)$ in Corollary 3.1.

Lemma 3.5 $(L, ({}^{-1}\theta)^*)$ is a Smarandache quasigroup with associative subquasigroup $(S, ({}^{-1}\theta)^*)$ if and only if any of the following equivalent statements is true.

1. $(S, ({}^{-1}\theta)^*)$ is isotopic to (S, θ).
2. $(S, {}^{-1}\theta)$ is isotopic to (S, θ^*).
3. $(S, ({}^{-1}\theta)^*)$ is isotopic to $(S, (\theta^{-1})^*)$.
4. $(S, {}^{-1}\theta)$ is isotopic to (S, θ^{-1}).

Proof. Replace (L, θ) with $(L, ({}^{-1}\theta)^*)$ in Theorem 3.1.

Corollary 3.6 $(L, ({}^{-1}\theta)^*)$ is a Smarandache quasigroup with associative subquasigroup $(S, ({}^{-1}\theta)^*)$ if and only if any of the six Khalil conditions is true for some four parastrophes of (S, θ).

Proof. Replace (L, θ) with $(L, ({}^{-1}\theta)^*)$ in Corollary 3.1.

Theorem 3.2 (L, π_i) is a Smarandache quasigroup with associative subquasigroup (S, π_i) $\forall\ i \in \{1, 2, 3, 4, 5, 6\}$ if and only if for any of some four $j \in \{1, 2, 3, 4, 5, 6\}$, (S, π_j) is an isotope of (S, π_i) or (S, π_k) for one $k \in \{1, 2, 3, 4, 5, 6\}$ such that $i \neq j \neq k$.

Proof. This is simply the summary of Theorem 3.1, Lemma 3.1, Lemma 3.2, Lemma 3.3, Lemma 3.4 and Lemma 3.5.

Corollary 3.7 (L, π_i) is a Smarandache quasigroup with associative subquasigroup (S, π_i)
$\forall\ i \in \{1, 2, 3, 4, 5, 6\}$ if and only if any of the six Khalil conditions is true for any of some four of (S, π_i).

Proof. This can be deduced from Theorem 3.2 and the Khalil conditions or by combining Corollary 3.1, Corollary 3.2, Corollary 3.3, Corollary 3.4, Corollary 3.5 and Corollary 3.6.

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Perfect powers in Smarandache n-Expressions

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Abstract In [1] I studied the concept of Smarandache n-expressions, for example I proposed formulas, found solutions, proposed open questions, and conjectured, but all for the fixed 3, and 2 numbers, but what will happen if these equations have different fixed numbers such as 7? This paper will answer this question.

Keywords Perfect powers in Smarandache N-Expressions, perfect squares, identity.

§1. Introduction

In [4] M. Perez and E. Burton, documented that J. Castillo [5], asked how many primes are there in the Smarandache n-expressions:

$$x_1^{x_2} + x_2^{x_3} + \cdots + x_n^{x_1},$$

where $n > 1$, $x_1, x_2, \cdots, x_n > 1$, and $(x_1, x_2, \cdots x_n) = 1$.

In this paper, with only slight modification of the above equation we got the following equation namely:

$$a^{x_1} + a^{x_2} + \cdots + a^{x_n},$$

where $a > 1$, $x_1, x_2, \cdots, x_n > 1$, and $(a, x_1, x_2, \cdots, x_n) = 1$.

§2.Main results and proofs

I will study the following cases of above equation.

Case 1. The solution of equation (1) is given by

$$7^p + 7^q + 7^r + 7^s = k^2, \tag{1}$$

where $p = 2m$, $q = 2m + 1$, $r = 2m + 2$, $s = 2m + 3$, and $k = 20 \cdot 7^m$.

Proof. Assume $k = 20 \cdot 7^m$, then $k^2 = 400 \cdot 7^2 m$, i.e.

$$k^2 = 400 \cdot 7^{2m} = (1 + 7 + 49 + 343)7^{2m} = 7^{2m} + 7^{2m+1} + 7^{2m+2} + 7^{2m+3}.$$

Hence $p = 2m$, $q = 2m + 1$, $r = 2m + 2$, and $s = 2m + 3$.

The first 11th solution of (1) is given in Table 1 below:

Table 1

m	$7^p + 7^q + 7^r + 7^s$	k^2
0	$7^0 + 7^1 + 7^2 + 7^3$	20^2
1	$7^2 + 7^3 + 7^4 + 7^5$	140^2
2	$7^4 + 7^5 + 7^6 + 7^7$	980^2
3	$7^6 + 7^7 + 7^8 + 7^9$	6860^2
4	$7^8 + 7^9 + 7^{10} + 7^{11}$	48020^2
5	$7^{10} + 7^{11} + 7^{12} + 7^{13}$	336140^2
6	$7^{12} + 7^{13} + 7^{14} + 7^{15}$	2352980^2
7	$7^{14} + 7^{15} + 7^{16} + 7^{17}$	16470860^2
8	$7^{16} + 7^{17} + 7^{18} + 7^{19}$	115296020^2
9	$7^{18} + 7^{19} + 7^{20} + 7^{21}$	807072140^2
10	$7^{20} + 7^{21} + 7^{22} + 7^{23}$	564950498^2

The first terms and the m-th terms of the sequence (the last column on Table 1 are:)

$$400, \quad 9600, \quad 960400, \quad 4705900, \quad \cdots, 20 \cdot 2 \cdot 7^{2m}, \cdots . \tag{2}$$

$$20, \quad 140, \quad 980, \quad 68600, \quad \cdots, 20 \cdot 7^{2m}, \cdots \tag{3}$$

I have noticed there is no prime numbers in geometric series (3), (excluding the prime 7). The

$$\sum_{i}^{m} 20 \cdot 7^{2m} = \frac{10(7^m - 1)}{3},$$

and there is no limit, since $\frac{10(7^m-1)}{3}$ becomes large as m approach infinity. The sequence has no limit, therefore it is divergent, but the summation of reciprocal convergent.

Equation (1) has the following important properties, i.e.

$$\begin{aligned} 7^p + 7^q + 7^r + 7^s &= a^2 - b^2 = c^2 - d^2 = e^2 - f^2 = g^2 - h^2 = i^2 - k^2 \\ &= j^2 - l^2 = o^2 - t^2 = u^2 - v^2 = w^2 - x^2. \end{aligned} \tag{4}$$

The solution of (4) are

$$a = 25 \cdot 7^{n-1}, \quad b = 15 \cdot 7^{n-1}, \quad c = 100 \cdot 7^{2n-2} + 1,$$

$$d = 100 \cdot 7^{2n-2} - 1, \quad e = 50 \cdot 7^{2n-2} + 2, \quad f = 50 \cdot 7^{2n-2} - 2,$$

$$g = 4 \cdot 7^{2n-2} + 25, \quad h = 4 \cdot 7^{2n-2} - 25, \quad i = 10 \cdot 7^{2n-1} + 10,$$

$$k = 10 \cdot 7^{2n-1} - 10, \quad j = 50 \cdot 7^{2n-1} + 20, \quad l = 50 \cdot 7^{2n-1} - 20,$$

$$o = 7^{2n-1} + 100, \quad t = 7^{2n-1} - 100, \quad u = 2 \cdot 7^{2n-1} + 50,$$

$$v = 2 \cdot 7^{2n-1} - 50, \quad w = 25 \cdot 7^{2n-1} + 4, \quad x = 25 \cdot 7^{2n-1} - 4.$$

For example, let $n = 2$ ($m = 1$), then we always have $n = m + 1$.

$$\begin{aligned} 175^2 - 105^2 &= 4901^2 - 4899^2 = 2452^2 - 2448^2 = 221^2 - 171^2 = 500^2 - 480^2 \\ &= 265^2 - 225^2 = 149^2 - 51^2 = 148^2 - 48^2 = 1229^2 - 1221^2 \\ &= 140^2 = 7^2 + 7^3 + 7^4 + 7^5. \end{aligned}$$

Conjecture 1. If p, q, r, s are distinct prime numbers, then the equation

$$7^p + 7^q + 7^r + 7^s = k^2,$$

will has no solution, (otherwise we have solutions in prime numbers, such as

$$7^2 + 7^2 + 7^2 + 7^2 = 14^2,$$

and

$$7^2 + 7^2 + 7^3 + 7^3 = 28^2.)$$

Case 2. The solution of equation (5) is given by

$$7^p + 7^q + 7^r + 7^s = k^2, \tag{5}$$

where $p = q = 2m$, $r = s = 2m + 1$, and $k = 4 \cdot 7^m$.

Proof. Assume $k = 4 \cdot 7^m$, then $k^2 = 16 \cdot 7^2 m$, i.e.

$$k^2 = 16 \cdot 7^{2m} = (1 + 1 + 7 + 7)7^{2m} = 7^{2m} + 7^{2m} + 7^{2m+1} + 7^{2m+1}.$$

Hence $p = q = 2m$, $r = s = 2m + 1$.

The first 11th solution of (5) is given in Table 2 below:

Table 2

m	$7^p + 7^q + 7^r + 7^s$	k^2
0	$7^0 + 7^0 + 7^1 + 7^1$	4^2
1	$7^2 + 7^2 + 7^3 + 7^3$	28^2
2	$7^4 + 7^4 + 7^5 + 7^5$	196^2
3	$7^6 + 7^6 + 7^7 + 7^7$	1372^2
4	$7^8 + 7^8 + 7^9 + 7^9$	9604^2
5	$7^{10} + 7^{10} + 7^{11} + 7^{11}$	67228^2
6	$7^{12} + 7^{12} + 7^{13} + 7^{13}$	470596^2
7	$7^{14} + 7^{14} + 7^{15} + 7^{15}$	3294172^2
8	$7^{16} + 7^{16} + 7^{17} + 7^{17}$	23059204^2
9	$7^{18} + 7^{18} + 7^{19} + 7^{19}$	161414428^2
10	$7^{20} + 7^{20} + 7^{21} + 7^{21}$	1129900996^2

Equation (5) has the following important properties, i.e.

$$7^p + 7^q + 7^r + 7^s = a^2 - b^2 = c^2 - d^2 = e^2 - f^2 = g^2 - h^2, \tag{6}$$

where the solution of (6) are

$$a = 5 \cdot 7^{n-1}, \quad b = 3 \cdot 7^n, \qquad c = 4 \cdot 7^{2n-2} + 1, \quad d = 4 \cdot 7^{2n-2} - 1,$$

$$e = 2 \cdot 7^{2n-2} + 2, \ f = 2 \cdot 7^{2n-2} - 2, \ g = 7^{2n-2} + 4, \ h = 7^{2n-2} - 4.$$

For example, let $n = 2$ ($m = 1$), then we have

$$\begin{aligned} 35^2 - 212 &= 197^2 - 195^2 = 100^2 - 96^2 = 53^2 - 45^2 \\ &= 28^2 = 7^2 + 7^2 + 7^3 + 7^3. \end{aligned}$$

Case 3. The solution of equation (7) is given by

$$7^p + 7^q + 7^r + 7^s = k^2, \tag{7}$$

where $p = q = 2m$, $r = s = 2m + 2$, and $k = 10 \cdot 7^m$.

Proof. Assume $k = 10 \cdot 7^m$, then $k^2 = 100 \cdot 7^2 m$, i.e.

$$k^2 = 100 \cdot 7^{2m} = (1 + 1 + 49 + 49)7^{2m} = 7^{2m} + 7^{2m} + 7^{2m+2} + 7^{2m+2}.$$

Hence $p = q = 2m$, $r = s = 2m + 2$.

The first 11th solution of (7) is given in Table 3 below:

Table 3

m	$7^p + 7^q + 7^r + 7^s$	k^2
0	$7^0 + 7^0 + 7^2 + 7^2$	10^2
1	$7^2 + 7^2 + 7^4 + 7^4$	70^2
2	$7^4 + 7^4 + 7^6 + 7^6$	490^2
3	$7^6 + 7^6 + 7^8 + 7^8$	3430^2
4	$7^8 + 7^8 + 7^{10} + 7^{10}$	24010^2
5	$7^{10} + 7^{10} + 7^{12} + 7^{12}$	168070^2
6	$7^{12} + 7^{12} + 7^{14} + 7^{14}$	1176490^2
7	$7^{14} + 7^{14} + 7^{16} + 7^{16}$	8235430^2
8	$7^{16} + 7^{16} + 7^{18} + 7^{18}$	57648010^2
9	$7^{18} + 7^{18} + 7^{20} + 7^{20}$	403536070^2
10	$7^{20} + 7^{20} + 7^{22} + 7^{22}$	2824752490^2

Equation (7) has the following important properties, i.e.

$$7^p + 7^q + 7^r + 7^s = k^2 = a^2 + b^2, \tag{8}$$

where the solution of (8) are

$$a = 8 \cdot 7^{n-1}, \quad b = 6 \cdot 7^{n-1}.$$

For example, let $n = 2$ ($m = 1$), then we have

$$7^2 + 7^2 + 7^4 + 7^4 = 56^2 + 42^2 = 70^2.$$

Case 4. The solution of equation (9) is given by

$$7^p + 7^q + 7^r + 7^s = k^2, \tag{9}$$

where $p = q = r = s = 2m$, and $k = 2 \cdot 7^m$.

Proof. Assume $k = 2 \cdot 7^m$, then $k^2 = 4 \cdot 7^2 m$, i.e.

$$k^2 = 4 \cdot 7^{2m} = (1 + 1 + 1 + 1)7^{2m} = 7^{2m} + 7^{2m} + 7^{2m+2} + 7^{2m+2}.$$

Hence $p = q = r = s = 2m$.

The first 11th solution of (9) is given in Table 4 below:

Table 4

m	$7^p + 7^q + 7^r + 7^s$	k^2
0	$7^0 + 7^0 + 7^0 + 7^0$	2^2
1	$7^2 + 7^2 + 7^2 + 7^2$	14^2
2	$7^4 + 7^4 + 7^4 + 7^4$	98^2
3	$7^6 + 7^6 + 7^6 + 7^6$	686^2
4	$7^8 + 7^8 + 7^8 + 7^8$	4802^2
5	$7^{10} + 7^{10} + 7^{10} + 7^{10}$	33614^2
6	$7^{12} + 7^{12} + 7^{12} + 7^{12}$	235298^2
7	$7^{14} + 7^{14} + 7^{14} + 7^{14}$	1647068^2
8	$7^{16} + 7^{16} + 7^{16} + 7^{16}$	11529602^2
9	$7^{18} + 7^{18} + 7^{18} + 7^{18}$	80707214^2
10	$7^{20} + 7^{20} + 7^{20} + 7^{20}$	564950498^2

Equation (7) has the following important properties, i.e.

$$7^p + 7^q + 7^r + 7^s = k_i^2 + k_{i+1}^2 = a^2 + a^2. \tag{10}$$

where $k_i = 2 \cdot 7^m$ and $a = 10 \cdot 7^m$.

Examples of equation (10):

1) $2^2 + 4^2 = 10^2 + 10^2$, (divided both sides by 2 given $1^2 + 7^2 = 5^2 + 5^2$)

2) $14^2 + 98^2 = 70^2 + 70^2$, (divided both sides by 14 given $1^2 + 7^2 = 5^2 + 5^2$)

3) $98^2 + 686^2 = 490^2 + 490^2$. (divided both sides by 98 given $1^2 + 7^2 = 5^2 + 5^2$)

These examples suggested a formula that gives three perfect squares which are in the arithmetic progression. So for the positive m and n with $m > n$, put

$$x = 2mn - m^2 + n^2,$$

$$y = m^2 + n^2,$$

$$z = 2mn + m^2 - n^2,$$

such as

$$y^2 + y^2 = z^2 + x^2.$$

So if $m = 2$, $n = 1$, then we will have

$$1^2 + 7^2 = 5^2 + 5^2.$$

Case 5. The solution of equation (11) is given by

$$7^p + 7^q + 7^r + 7^s = 2^4 \cdot 5^2 \cdot 7^p, \tag{11}$$

where $p = 2m + 1$, $q = 2m + 2$, $r = 2m + 3$, and $s = 2m + 4$.

Proof. Assume $2^4 \cdot 5^2 \cdot 7^{2m+1}$, is the sum of equation (11), then

$$2^4 \cdot 5^2 \cdot 7^{2m+1} = (1 + 7 + 49 + 343)7^{2m+1} = 7^{2m+1} + 7^{2m+2} + 7^{2m+3} + 7^{2m+4}.$$

Hence $p = 2m + 1$, $q = 2m + 2$, $r = 2m + 3$, $s = 2m + 4$.

The first 7th solution is given in Table 5 below:

Table 5

m	$7^p + 7^q + 7^r + 7^s$	$2^4 \cdot 5^2 \cdot 7^p$
0	$7^1 + 7^2 + 7^3 + 7^4$	$2^4 \cdot 5^2 \cdot 7^1$
1	$7^3 + 7^4 + 7^5 + 7^6$	$2^4 \cdot 5^2 \cdot 7^3$
2	$7^5 + 7^6 + 7^7 + 7^8$	$2^4 \cdot 5^2 \cdot 7^5$
3	$7^7 + 7^8 + 7^9 + 7^{10}$	$2^4 \cdot 5^2 \cdot 7^7$
4	$7^9 + 7^{10} + 7^{11} + 7^{12}$	$2^4 \cdot 5^2 \cdot 7^9$
5	$7^{11} + 7^{12} + 7^{13} + 7^{14}$	$2^4 \cdot 5^2 \cdot 7^{11}$
6	$7^{13} + 7^{14} + 7^{15} + 7^{16}$	$2^4 \cdot 5^2 \cdot 7^{13}$

The first terms and the m-th terms of the sequence (the last column on Table 5 are:)

$$2^4 \cdot 5^2 \cdot 7^1, \quad 2^4 \cdot 5^2 \cdot 7^3, \quad 2^4 \cdot 5^2 \cdot 7^5, \quad \cdots, \quad 2^4 \cdot 5^2 \cdot 7^{2m+1}, \cdots, \tag{12}$$

where the square roots are

$$20 \cdot 7^{1/2}, \quad 20 \cdot 7^{3/2}, \quad 20 \cdot 7^{5/2}, \quad 20 \cdot 7^{7/2}, \quad \cdots, \quad 20 \cdot 7^{2m+1/2}, \cdots. \tag{13}$$

Notice that there is no prime numbers in geometric series (13).

The

$$\sum_{i}^{m} 20 \cdot 7^{2i+1/2} = \frac{10 \cdot 7^{1/2} \cdot 7^{2m+1} - 1}{3},$$

and there is no limit, since $\frac{10 \cdot 7^{1/2} \cdot 7^{2m+1} - 1}{3}$ becomes large as m approach infinity. The sequence has no limit, therefore it is divergent, but the summation of reciprocal convergent.

Conjecture 2. If p, q, r, s are distinct prime numbers, then the equation

$$7^p + 7^q + 7^r + 7^s = 2^4 \cdot 5^2 \cdot 7^p,$$

has no solution.

Case 6. The solution of equation (11) is given by

$$7^p + 7^q + 7^r + 7^s = 2^4 \cdot 7^p, \tag{14}$$

where $p = q = 2m + 1$, $r = s = 2m + 2$.

Proof. Assume $2^4 \cdot 7^{2m+1}$ is the sum of equation (14), then

$$2^4 \cdot 7^{2m+1} = (1 + 1 + 7 + 7)7^{2m+1} = 7^{2m+1} + 7^{2m+1} + 7^{2m+2} + 7^{2m+2}.$$

Hence $p = q = 2m + 1$, $r = s = 2m + 2$.

The first 11th solution is given in Table 6 below:

Table 6

m	$7^p + 7^q + 7^r + 7^s$	$2^4 \cdot 7^p$
0	$7^1 + 7^1 + 7^2 + 7^2$	$2^4 \cdot 7^1$
1	$7^3 + 7^3 + 7^4 + 7^4$	$2^4 \cdot 7^3$
2	$7^5 + 7^5 + 7^6 + 7^6$	$2^4 \cdot 7^5$
3	$7^7 + 7^7 + 7^8 + 7^8$	$2^4 \cdot 7^7$
4	$7^9 + 7^9 + 7^{10} + 7^{10}$	$2^4 \cdot 7^9$
5	$7^{11} + 7^{11} + 7^{12} + 7^{12}$	$2^4 \cdot 7^{11}$
6	$7^{13} + 7^{13} + 7^{14} + 7^{14}$	$2^4 \cdot 7^{13}$
7	$7^{15} + 7^{15} + 7^{16} + 7^{16}$	$2^4 \cdot 7^{15}$
8	$7^{17} + 7^{17} + 7^{18} + 7^{18}$	$2^4 \cdot 7^{17}$
9	$7^{19} + 7^{19} + 7^{20} + 7^{20}$	$2^4 \cdot 7^{19}$
10	$7^{21} + 7^{21} + 7^{22} + 7^{22}$	$2^4 \cdot 7^{21}$

If we look deeply in equation (11), (14), we can find the following relation

$$20^4 - 2^4 = 97^2 - 95^2 = 50^2 - 46^2 = 28^2 - 20^2.$$

Case 7. The solution of equation (15) is given by

$$7^p + 7^q + 7^r + 7^s = 2^2 \cdot 5^2 \cdot 7^p, \tag{15}$$

where $p = q = 2m + 1$, $r = s = 2m + 3$.

Proof. Assume $2^2 \cdot 5^2 \cdot 7^{2m+1}$, is the sum of equation (15), then

$$2^2 \cdot 5^2 \cdot 7^{2m+1} = (1 + 1 + 49 + 49)7^{2m+1} = 7^{2m+1} + 7^{2m+1} + 7^{2m+3} + 7^{2m+3}.$$

Hence $p = q = 2m + 1$, $r = s = 2m + 2$.

The first 11th solution of (15) is given in Table 7 below:

Table 7

m	$7^p + 7^q + 7^r + 7^s$	$2^2 \cdot 5^2 \cdot 7^p$
0	$7^1 + 7^1 + 7^3 + 7^3$	$2^2 \cdot 5^2 \cdot 7^1$
1	$7^3 + 7^3 + 7^5 + 7^5$	$2^2 \cdot 5^2 \cdot 7^3$
2	$7^5 + 7^5 + 7^7 + 7^7$	$2^2 \cdot 5^2 \cdot 7^5$
3	$7^7 + 7^7 + 7^9 + 7^9$	$2^2 \cdot 5^2 \cdot 7^7$
4	$7^9 + 7^9 + 7^{11} + 7^{11}$	$2^2 \cdot 5^2 \cdot 7^9$
5	$7^{11} + 7^{11} + 7^{13} + 7^{13}$	$2^2 \cdot 5^2 \cdot 7^{11}$
6	$7^{13} + 7^{13} + 7^{15} + 7^{15}$	$2^2 \cdot 5^2 \cdot 7^{13}$
7	$7^{15} + 7^{15} + 7^{17} + 7^{17}$	$2^2 \cdot 5^2 \cdot 7^{15}$
8	$7^{17} + 7^{17} + 7^{19} + 7^{19}$	$2^2 \cdot 5^2 \cdot 7^{17}$
9	$7^{19} + 7^{19} + 7^{21} + 7^{21}$	$2^2 \cdot 5^2 \cdot 7^{19}$
10	$7^{21} + 7^{21} + 7^{23} + 7^{23}$	$2^2 \cdot 5^2 \cdot 7^{21}$

case 8. The solution of equation (16) is given by

$$7^p + 7^q + 7^r + 7^s = 1201 \cdot 2^2 \cdot 5^2 \cdot 7^p, \tag{16}$$

where $p = 2m$, $q = 2m + 2$, $r = 2m + 4$, $s = 2m + 6$.

Proof. Assume $1201 \cdot 2^2 \cdot 5^2 \cdot 7^{2m+1}$, is the sum of equation (16), then

$$1201 \cdot 2^2 \cdot 5^2 \cdot 7^{2m+1} = (1 + 49 + 2401 + 117649)7^{2m} = 7^{2m} + 7^{2m+2} + 7^{2m+4} + 7^{2m+6}.$$

Hence $p = 2m$, $q = 2m + 2$, $r = 2m + 4$, $s = 2m + 6$.

The first 11th solution of (16) is given in Table 8 below:

Table 8

m	$7^p + 7^q + 7^r + 7^s$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^p$
0	$7^0 + 7^2 + 7^4 + 7^6$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^0$
1	$7^2 + 7^4 + 7^6 + 7^8$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^2$
2	$7^4 + 7^6 + 7^8 + 7^{10}$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^4$
3	$7^6 + 7^8 + 7^{10} + 7^{12}$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^6$
4	$7^8 + 7^{10} + 7^{12} + 7^{14}$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^8$
5	$7^{10} + 7^{12} + 7^{14} + 7^{16}$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^{10}$
6	$7^{12} + 7^{14} + 7^{16} + 7^{18}$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^{12}$
7	$7^{14} + 7^{16} + 7^{18} + 7^{20}$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^{14}$
8	$7^{16} + 7^{18} + 7^{20} + 7^{22}$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^{16}$
9	$7^{18} + 7^{20} + 7^{22} + 7^{24}$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^{18}$
10	$7^{20} + 7^{22} + 7^{24} + 7^{26}$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^{20}$

Case 9. The solution of equation (17) is given by

$$7^p + 7^q + 7^r + 7^s = 1201 \cdot 2^2 \cdot 5^2 \cdot 7^p, \tag{17}$$

where $p = 2m + 1$, $q = 2m + 3$, $r = 2m + 5$, $s = 2m + 7$.

Proof. Assume $1201 \cdot 2^2 \cdot 5^2 \cdot 7^{2m+1}$, is the sum of equation (17), then

$$1201 \cdot 2^2 \cdot 5^2 \cdot 7^{2m+1} = (1 + 49 + 2401 + 117649)7^{2m+1} = 7^{2m+1} + 7^{2m+3} + 7^{2m+5} + 7^{2m+7}.$$

Hence $p = 2m + 1$, $q = 2m + 3$, $r = 2m + 5$, $s = 2m + 7$.

The first 11th solution of (17) is given in Table 9 below:

Table 9

m	$7^p + 7^q + 7^r + 7^s$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^p$
0	$7^1 + 7^3 + 7^5 + 7^7$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^1$
1	$7^3 + 7^5 + 7^7 + 7^9$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^3$
2	$7^5 + 7^7 + 7^9 + 7^{11}$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^5$
3	$7^7 + 7^9 + 7^{11} + 7^{13}$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^7$
4	$7^9 + 7^{11} + 7^{13} + 7^{15}$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^9$
5	$7^{11} + 7^{13} + 7^{15} + 7^{17}$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^{11}$
6	$7^{13} + 7^{15} + 7^{17} + 7^{19}$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^{13}$
7	$7^{15} + 7^{17} + 7^{19} + 7^{21}$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^{15}$
8	$7^{17} + 7^{19} + 7^{21} + 7^{23}$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^{17}$
9	$7^{19} + 7^{21} + 7^{23} + 7^{25}$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^{19}$
10	$7^{21} + 7^{23} + 7^{25} + 7^{27}$	$1201 \cdot 2^2 \cdot 5^2 \cdot 7^{21}$

Case 10. The solution of equation (18) is given by

$$7^p + 7^q + 7^r + 7^s + 7^t + 7^u + 7^v + 7^w + 7^z = 1201 \cdot 2^5 \cdot 5^2 \cdot 7^p, \tag{18}$$

where $p = m$, $q = m + 1$, $r = m + 2$, $s = m + 3$, $t = m + 4$, $u = m + 5$, $w = m + 6$, $z = m + 7$.

Proof. Assume $1201 \cdot 2^2 \cdot 5^2 \cdot 7^{2m+1}$, is the sum of equation (18), then

$$1201 \cdot 2^2 \cdot 5^2 \cdot 7^{2m+1} = (1 + 49 + 2401 + 117649)7^{2m+1} = 7^{2m+1} + 7^{2m+3} + 7^{2m+5} + 7^{2m+7}.$$

Hence $p = m$, $q = m + 1$, $r = m + 2$, $s = m + 3$, $t = m + 4$, $u = m + 5$, $w = m + 6$, $z = m + 7$.

The first 11th solution of (18) is given in Table 10 below:

Table 10

m	$7^p+7^q+7^r+7^s+7^t+7^u+7^w+7^z$	$1201\cdot 2^5\cdot 5^2\cdot 7^p$
0	$7^0+7^1+7^2+7^3+7^4+7^5+7^6+7^7$	$1201\cdot 2^5\cdot 5^2\cdot 7^0$
1	$7^1+7^2+7^3+7^4+7^5+7^6+7^7+7^8$	$1201\cdot 2^5\cdot 5^2\cdot 7^1$
2	$7^2+7^3+7^4+7^5+7^6+7^7+7^8+7^9$	$1201\cdot 2^5\cdot 5^2\cdot 7^2$
3	$7^3+7^4+7^5+7^6+7^7+7^8+7^9+7^{10}$	$1201\cdot 2^5\cdot 5^2\cdot 7^3$
4	$7^4+7^5+7^6+7^7+7^8+7^9+7^{10}+7^{11}$	$1201\cdot 2^5\cdot 5^2\cdot 7^4$
5	$7^5+7^6+7^7+7^8+7^9+7^{10}+7^{11}+7^{12}$	$1201\cdot 2^5\cdot 5^2\cdot 7^5$
6	$7^6+7^7+7^8+7^9+7^{10}+7^{11}+7^{12}+7^{13}$	$1201\cdot 2^5\cdot 5^2\cdot 7^6$
7	$7^7+7^8+7^9+7^{10}+7^{11}+7^{12}+7^{13}+7^{14}$	$1201\cdot 2^5\cdot 5^2\cdot 7^7$
8	$7^8+7^9+7^{10}+7^{11}+7^{12}+7^{13}+7^{14}+7^{15}$	$1201\cdot 2^5\cdot 5^2\cdot 7^8$
9	$7^9+7^{10}+7^{11}+7^{12}+7^{13}+7^{14}+7^{15}+7^{16}$	$1201\cdot 2^5\cdot 5^2\cdot 7^9$
10	$7^{10}+7^{11}+7^{12}+7^{13}+7^{14}+7^{15}+7^{16}+7^{17}$	$1201\cdot 2^5\cdot 5^2\cdot 7^{10}$

Case 11. The solution of equation (19) is given by

$$7^p+7^q+7^r+7^s+7^t+7^u+7^v+7^w+7^z=k^3, \tag{19}$$

where $p=q=r=s=t=u=w=z=3m$.

Proof. Assume $k=2\cdot 7^m$, then $k^3=2^3\cdot 7^{3m}$, i.e.

$$\begin{aligned} k^2 &= 8\cdot 7^{3m}=(1+1+1+1+1+1+1+1)7^{3m} \\ &= 7^{3m}+7^{3m}+7^{3m}+7^{3m}+7^{3m}+7^{3m}+7^{3m}+7^{3m}. \end{aligned}$$

Hence $p=q=r=s=t=u=w=z=3m$.

The first 11th solution of (19) is given in Table 11 below:

Table 11

m	$7^p+7^q+7^r+7^s+7^t+7^u+7^w+7^z$	$k^3=2^3\cdot 7^{3m}$
0	$7^0+7^0+7^0+7^0+7^0+7^0+7^0+7^0$	2^3
1	$7^3+7^3+7^3+7^3+7^3+7^3+7^3+7^3$	14^3
2	$7^6+7^6+7^6+7^6+7^6+7^6+7^6+7^6$	98^3
3	$7^9+7^9+7^9+7^9+7^9+7^9+7^9+7^9$	686^3
4	$7^{12}+7^{12}+7^{12}+7^{12}+7^{12}+7^{12}+7^{12}+7^{12}$	4802^3
5	$7^{15}+7^{15}+7^{15}+7^{15}+7^{15}+7^{15}+7^{15}+7^{15}$	33614^3
6	$7^{18}+7^{18}+7^{18}+7^{18}+7^{18}+7^{18}+7^{18}+7^{18}$	235298^3
7	$7^{21}+7^{21}+7^{21}+7^{21}+7^{21}+7^{21}+7^{21}+7^{21}$	1647068^3
8	$7^{24}+7^{24}+7^{24}+7^{24}+7^{24}+7^{24}+7^{24}+7^{24}$	11529602^3
9	$7^{27}+7^{27}+7^{27}+7^{27}+7^{27}+7^{27}+7^{27}+7^{27}$	80707214^3
10	$7^{30}+7^{30}+7^{30}+7^{30}+7^{30}+7^{30}+7^{30}+7^{30}$	5649504983

I find interesting identity like this:

1) $14^3 - 2^3 = 685^2 - 683^2 = 344^2 - 340^2 = 175^2 - 167^2$,

2) $98^3 - 14^3 = 234613^2 - 234611^2 = 117308^2 - 117304^2 = 58657^2 - 58649^2$, and so on.

§3. Open questions

The following equations have infinitely many solutions, find them.

1) $-7^p + 7^q - 7^r + 7^s = 3 \cdot 2^2 \cdot 5^2 \cdot 7^p$ (has two different solutions),

2) $7^p + 7^q = 2^3 \cdot 7^p$,

3) $7^p + 7^q = 2 \cdot 5^2 \cdot 7^p$.

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Palindromic permutations and generalized Smarandache palindromic permutations

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Abstract The idea of left(right) palindromic permutations(LPPs, RPPs) and left(right) generalized Smarandache palindromic permutations(LGSPPs, RGSPPs) are introduced in symmetric groups S_n of degree n. It is shown that in S_n, there exist a LPP and a RPP and they are unique(this fact is demonstrated using S_2 and S_3). The dihedral group D_n is shown to be generated by a RGSPP and a LGSPP(this is observed to be true in S_3) but the geometric interpretations of a RGSPP and a LGSPP are found not to be rotation and reflection respectively. In S_3, each permutation is at least a RGSPP or a LGSPP. There are 4 RGSPPs and 4 LGSPPs in S_3, while 2 permutations are both RGSPPs and LGSPPs. A permutation in S_n is shown to be a LPP or RPP(LGSPP or RGSPP) if and only if its inverse is a LPP or RPP(LGSPP or RGSPP) respectively. Problems for future studies are raised.

Keywords Permutation, symmetric groups, palindromic permutations, generalized Smarandache palindromic permutations.

§1. Introduction

According to Ashbacher and Neirynck [1], an integer is said to be a palindrome if it reads the same forwards and backwards. For example, 12321 is a palindromic number. They also stated that it is easy to prove that the density of the palindromes is zero in the set of positive integers and they went ahead to answer the question on the density of generalized Smarandache palindromes (GSPs) by showing that the density of GSPs in the positive integers is approximately 0.11. Gregory [2], Smarandache [8] and Ramsharan [7] defined generalized Smarandache palindrome (GSP) as any integer or number of the form

$a_1a_2a_3\cdots a_na_n\cdots a_3a_2a_1$ or $a_1a_2a_3\cdots a_{n-1}a_na_{n-1}\cdots a_3a_2a_1$

where all a_1, a_2, a_3, $\cdots$, $a_n \in \mathbb{N}$ having one or more digits. On the other hand, Hu [3] calls any integer or number of this form a Smarandache generalized palindrome (SGP). His naming will not be used here the first naming will be adopted. Numbers of this form have also been considered by Khoshnevisan [4], [5] and [6]. For the sake of clari ¯ cation, it must be mentioned that the possibility of the trivial case of enclosing the entire number is excluded. For example, 12345 can be written as (12345). In this case, the number is simply said to be a palindrome or a palindromic number as it was mentioned earlier on. So, every number is a GSP. But this possibility is eliminated by requiring that each number be split into at least two segments if it is not a regular palindrome. Trivially, since each regular palindrome is also

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a GSP and there are GSPs that are not regular palindromes, there are more GSPs than there are regular palindromes. As mentioned by Gregory [2], very interesting GSPs are formed from smarandacheian sequences. For an illustration he cited the smarandacheian sequence

$$11, 1221, 123321, \ldots, 123456789987654321, 1234567891010987654321,$$

$$12345678910111110987654321, \ldots$$

and observed that all terms are all GSPs. He also mentioned that it has been proved that the GSP 1234567891010987654321 is a prime and concluded his work by possing the question of How many primes are in the GSP sequence above?

Special mappings such as morphisms(homomorphisms, endomorphisms, automorphisms, isomorphisms e.t.c) have been useful in the study of the properties of most algebraic structures(e.g groupoids, quasigroups, loops, semigroups, groups e.tc.). In this work, the notion of palindromic permutations and generalized Smarandache palindromic permutations are introduced and studied using the symmetric group on the set $\mathbb{N}$ and this can now be viewed as the study of some palindromes and generalized Smarandache palindromes of numbers.

The idea of left(right) palindromic permutations(LPPs, RPPs) and left(right) generalized Smarandache palindromic permutations(LGSPPs, RGSPPs) are introduced in symmetric groups S_n of degree n. It is shown that in S_n, there exist a LPP and a RPP and they are unique. The dihedral group D_n is shown to be generated by a RGSPP and a LGSPP but the geometric interpretations of a RGSPP and a LGSPP are found not to be rotation and reflection respectively. In S_3, each permutation is at least a RGSPP or a LGSPP. There are 4 RGSPPs and 4 LGSPPs in S_3, while 2 permutations are both RGSPPs and LGSPPs. A permutation in S_n is shown to be a LPP or RPP(LGSPP or RGSPP) if and only if its inverse is a LPP or RPP(LGSPP or RGSPP) respectively. Some of these results are demonstrated with S_2 and S_3. Problems for future studies are raised.

But before then, some definitions and basic results on symmetric groups in classical group theory which shall be employed and used are highlighted first.

§2. Preliminaries

Definition 2.1 Let X be a non-empty set. The group of all permutations of X under composition of mappings is called the symmetric group on X and is denoted by S_X. A subgroup of S_X is called a permutation group on X.

It is easily seen that a bijection $X \simeq Y$induces in a natural way an isomorphism $S_X \cong S_Y$. If $|X| = n$, S_X is denoted by S_n and called the symmetric group of degree n.

A permutation $\sigma \in S_n$ can be exhibited in the form

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}$$

consisting of two rows of integers; the top row has integers $1, 2, \ldots, n$, usually (but not necessarily) in their natural order, and the bottom row has $\sigma(i)$ below i for each $i = 1, 2, \ldots, n$.

This is called a two-row notation for a permutation. There is a simpler, one-row notation for a special kind of permutation called cycle.

Definition 2.2 Let $\sigma \in S_n$. If there exists a list of distinct integers $x_1, \ldots, x_r \in \mathbb{N}$ such that

$$\begin{aligned} &\sigma(x_i) = x_{i+1}, \qquad i = 1, \ldots, r-1, \\ &\sigma(x_r) = x_1, \\ &\sigma(x) = x \text{ if } x \notin \{x_1, \cdots, x_r\}, \end{aligned}$$

then σ is called a cycle of length r and denoted by $(x_1 \ldots x_r)$.

Remark 2.1 A cycle of length 2 is called a transposition. In other words , a cycle $(x_1 \ldots x_r)$ moves the integers $(x_1 \ldots x_r)$ one step around a circle and leaves every other integer in $\mathbb{N}$. If $\sigma(x) = x$, we say σ does not move x. Trivially, any cycle of length 1 is the identity mapping I or e. Note that the one-row notation for a cycle does not indicate the degree n, which has to be understood from the context.

Definition 2.3 Let X be a set of points in space, so that the distance $d(x, y)$ between points x and y is given for all $x, y \in X$. A permutation σ of X is called a symmetry of X if

$$d(\sigma(x), \sigma(y)) = d(x, y) \ \forall \ x, y \in X.$$

Let X be the set of points on the vertices of a regular polygon which are labelled $1, 2, \cdots, n$ i.e $X = \{1, 2, \cdots, n\}$.

The group of symmetries of a regular polygon P_n of n sides is called the dihedral group of degree n and denoted D_n.

Remark 2.2 It must be noted that D_n is a subgroup of S_n i.e $D_n \leq S_n$.

Definition 2.4 Let S_n be a symmetric group of degree n. If $\sigma \in S_n$ such that

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}$$

then

1. the number $N_\lambda(\sigma) = 12 \ldots n\sigma(n) \ldots \sigma(1)$ is called the left palindromic value(LPV) of σ.

2. the number $N_\rho(\sigma) = 12 \ldots n\sigma(1) \ldots \sigma(n)$ is called the right palindromic value(RPV) of σ.

Definition 2.5 Let $\sigma \in S_X$ such that

$$\sigma = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ \sigma(x_1) & \sigma(x_2) & \cdots & \sigma(x_n) \end{pmatrix}$$

If $X = \mathbb{N}$, then

1. σ is called a left palindromic permutation(LPP) if and only if the number $N_\lambda(\sigma)$ is a palindrome.

$$PP_\lambda(S_X) = \{\sigma \in S_X : \sigma \text{ is a LPP } \}$$

2. σ is called a right palindromic permutation(RPP) if and only if the number $N_\rho(\sigma)$ is a palindrome.

$$PP_\rho(S_X) = \{\sigma \in S_X : \sigma \text{ is a RPP } \}$$

3. σ is called a palindromic permutation(PP) if and only if it is both a LPP and a RPP.

$$PP(S_X) = \{\sigma \in S_X : \sigma \text{ is a LPP and a } RPP\} = PP_\lambda(S_X) \cap PP_\rho(S_X)$$

Definition 2.6 Let $\sigma \in S_X$ such that

$$\sigma = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ \sigma(x_1) & \sigma(x_2) & \cdots & \sigma(x_n) \end{pmatrix}$$

If $X = \mathbb{N}$, then

1. σ is called a left generalized Smarandache palindromic permutation($LGSPP$) if and only if the number N_λ is a GSP.

$$GSPP_\lambda(S_X) = \{\sigma \in S_X : \sigma \text{ is a } LGSPP\}$$

2. σ is called a right generalized Smarandache palindromic permutation($RGSPP$) if and only if the number N_ρ is a GSP.

$$GSPP_\rho(S_X) = \{\sigma \in S_X : \sigma \text{ is a RGSPP}\}$$

3. σ is called a generalized Smarandache palindromic permutation($GSPP$) if and only if it is both a $LGSPP$ and a $RGSPP$.

$$GSPP(S_X) = \{\sigma \in S_X : \sigma \text{ is a LGSPP and a } RGSPP\} = GSPP_\lambda(S_X) \cap GSPP_\rho(S_X)$$

Theorem 2.1 (Cayley Theorem) Every group is isomorphic to a permutation group.

Theorem 2.2 The dihedral group D_n is a group of order $2n$ generated by two elements σ, τ satisfying $\sigma^n = e = \tau^2$ and $\tau\sigma = \sigma^{n-1}\tau$, where

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \end{pmatrix},$$

and

$$\tau = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & n & \cdots & 2 \end{pmatrix}.$$

§3. Main Results

Theorem 3.1 In any symmetric group S_n of degree n, there exists

1. a LPP and it is unique.

2. a RPP and it is unique.

But there does not exist a PP.

Proof Let $\sigma \in S_n$, then

$$\sigma = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ \sigma(x_1) & \sigma(x_2) & \cdots & \sigma(x_n) \end{pmatrix}.$$

1. When

$$\sigma(n) = n, \sigma(n-1) = n-1, \cdots, \sigma(2) = 2, \quad \sigma(1) = 1$$

then the number

$$N_\lambda(\sigma) = 12\cdots n\sigma(n)\cdots\sigma(2)\sigma(1) = 12\cdots nn\cdots 21$$

is a palindrome which implies $\sigma \in PP_\lambda(S_n)$. So there exist a LPP. The uniqueness is as follows. Observe that

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix} = I.$$

Since S_n is a group for all $n \in \mathbb{N}$ and I is the identity element (mapping), then it must be unique.

2. When

$$\sigma(1) = n, \sigma(2) = n-1, \cdots, \sigma(n-1) = 2, \sigma(n) = 1,$$

then the number

$$N_\rho(\sigma) = 12\cdots n\sigma(1)\cdots\sigma(n-1)\sigma(n) = 12\cdots nn\cdots 21$$

is a palindrome which implies $\sigma \in PP_\rho S_n$. So there exist a RPP. The uniqueness is as follows. If there exist two of such, say σ_1 and σ_2 in S_n, then

$$\sigma_1 = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma_1(1) & \sigma_1(2) & \cdots & \sigma_1(n) \end{pmatrix} \quad \text{and} \quad \sigma_2 = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma_1(1) & \sigma_1(2) & \cdots & \sigma_1(n) \end{pmatrix}$$

such that

$$N_\rho(\sigma_1) = 12\cdots n\sigma_1(1)\cdots\sigma_1(n-1)\sigma_1(n)$$

and

$$N_\rho(\sigma_2) = 12\cdots n\sigma_2(1)\cdots\sigma_2(n-1)\sigma_2(n)$$

are palindromes which implies

$$\sigma_1(1) = n, \sigma_1(2) = n-1, \cdots, \sigma_1(n-1) = 2, \sigma_1(n) = 1,$$

and

$$\sigma_2(1) = n, \sigma_2(2) = n-1, \cdots, \sigma_2(n-1) = 2, \sigma_2(n) = 1.$$

So, $\sigma_1 = \sigma_2$, thus σ is unique.

The proof of the last part is as follows. Let us assume by contradiction that there exists a PP $\sigma \in S_n$. Then if

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix},$$

$$N_\lambda(\sigma) = 12\cdots n\sigma(n)\cdots\sigma(2)\sigma(1)$$

and

$$N_\rho(\sigma) = 12\cdots n\sigma(1)\cdots\sigma(n-1)\sigma(n)$$

are palindromes. So that $\sigma \in S_n$ is a PP. Consequently,

$$n = \sigma(n) = 1, n-1 = \sigma(n-1) = 2, \cdots, 1 = \sigma(1) = n,$$

so that σ is not a bijection which means $\sigma \notin S_n$. This is a contradiction. Hence, no PP exist.

Example 3.1 Let us consider the symmetric group S_2 of degree 2. There are two permutations of the set $\{1, 2\}$ given by

$$I = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix},$$

and

$$\delta = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

$$N_\rho(I) = 1212 = (12)(12), N_\lambda(I) = 1221 \; or \; N_\lambda(I) = 1(22)1,$$

$$N_\rho(\delta) = 1221 \; or \; N_\rho(\delta) = (12)(21) \; and \; N_\lambda(\delta) = 1212 = (12)(12).$$

So, I and δ are both RGSPPs and LGSPPs which implies I and δ are GSPPs i.e $I, \delta \in GSPP_\rho(S_2)$ and $I, \delta \in GSPP_\rho(S_2) \Rightarrow I, \delta \in GSPP(S_2)$. Therefore, $GSPP(S_2) = S_2$. Furthermore, it can be seen that the result in Theorem 3.1 is true for S_2 because only I is a LPP and only δ is a RPP. There is no PP as the theorem says.

Example 3.2 Let us consider the symmetric group S_3 of degree 3. There are six permutations of the set {1,2,3} given by

$$e = I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix},$$

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix},$$

$$\sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix},$$

$$e = I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix},$$

$$\tau_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix},$$

$$\tau_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix},$$

and

$$\tau_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

As claimed in Theorem 3.1, the unique LPP in S_3 is I while the unique RPP in S_3 is τ_2. There is no PP as the theorem says.

Lemma 3.1 In S_3, the following are true.

1. At least $\sigma \in GSPP_\rho(S_3)$ or $\sigma \in GSPP_\lambda(S_3)$ $\forall\, \sigma \in S_3$.
2. $|GSPP_\rho(S_3)| = 4$, $|GSPP_\lambda(S_3)| = 4$ and $|GSPP(S_3)| = 2$.

Proof Observe the following:

$$N_\lambda(I) = 123321,\ N_\rho(I) = 123123 = (123)(123).$$

$$N_\lambda(\sigma_1) = 123132, N_\rho(\sigma_1) = 123231 = 1(23)(23)1.$$

$$N_\lambda(\sigma_2) = 123213, N_\rho(\sigma_2) = 123312 = (12)(33)(12).$$

$$N_\lambda(\tau_1) = 123231 = 1(23)(23)1, N_\rho(\tau_1) = 123132.$$

$$N_\lambda(\tau_2) = 123123 = (123)(123), N_\rho(\tau_2) = 123321.$$

$$N_\lambda(\tau_3) = 123312 = (12)(33)(12), N_\rho(\tau_3) = 123213.$$

So, $GSPP_\lambda(S_3) = \{I, \tau_1, \tau_2, \tau_3\}$ and $GSPP_\rho(S_3) = \{I, \sigma_1, \sigma_2, \tau_2\}$. Thus, 1. is true .Therefore, $|GSPP_\rho(S_3)| = 4$, $|GSPP_\lambda(S_3)| = 4$ and $|GSPP(S_3)| = |GSPP_\rho(S_3) \bigcap GSPP_\lambda(S_3)| = 2$. So, 2. is true.

Lemma 3.2 S_3 is generated by a RGSPP and a LGSPP.

Proof Recall from Example 3.2 that

$$S_3 = \{I, e, \sigma_1, \sigma_2, \tau_1, \tau_2, \tau_3\}.$$

If $\sigma = \sigma_1$ and $\tau = \tau_1$, then it is easy to verify that

$$\sigma^2 = \sigma_2, \sigma^3 = e, \tau^2 = e, \sigma\tau = \tau_3, \sigma^2\tau = \tau_2 = \tau\sigma \; hence,$$

$$S_3 = \{e, \sigma, \sigma^2, \tau, \sigma\tau, \sigma^2\tau_3\} \Rightarrow S_3 = \langle \sigma, \tau \rangle.$$

From the proof Lemma 3.1, σ is a RGSPP and τ is a LGSPP. This justifies the claim.

Remark 3.1 In Lemma 3.2, S_3 is generated by a RGSPP and a LGSPP. Could this statement be true for all S_n of degree n? Or could it be true for some subgroups of S_n? Also, it is interesting to know the geometric meaning of a RGSPP and a LGSPP. So two questions are possed and the two are answered.

Question 3.1 1. Is the symmetric group S_n of degree n generated by a RGSPP and a LGSPP? If not, what permutation group (s) is generated by a RGSPP and a LGSPP?

2. Are the geometric interpretations of a RGSPP and a LGSPP rotation and reflection respectively?

Theorem 3.2 The dihedral group D_n is generated by a RGSPP and a LGSPP i.e $D_n = \langle \sigma, \tau \rangle$ where $\sigma \in GSPP_\rho(S_n)$ and $\tau \in GSPP_\lambda(S_n)$.

Proof Recall from Theorem 2.2 that the dihedral group $D_n = \langle \sigma, \tau \rangle$ where

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \end{pmatrix} = \begin{pmatrix} 1 & 2 & \cdots & n \\ 2 & 3 & \cdots & 1 \end{pmatrix}$$

and

$$\tau = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & n & \cdots & 2 \end{pmatrix}.$$

Observe that

$$N_\rho(\sigma) = 123\cdots n23\cdots n1 = 1(23\cdots n)(23\cdots n)1,\ N_\lambda(\sigma) = 123\cdots n1n\cdots 32.$$

$$N_\rho(\tau) = 12\cdots n1n\cdots 2,\ N_\lambda(\tau) = 12\cdots n2\cdots n1 = 1(2\cdots n)(2\cdots n)1.$$

So, $\sigma \in GSPP_\rho(S_n)$ and $\tau \in GSPP_\lambda(S_n)$. Therefore, the dihedral group D_n is generated by a RGSPP and a LGSPP.

Remark 3.2 In Lemma 3.2, it was shown that S_3 is generated by a RGSPP and a LGSPP. Considering Theorem 3.2 when $n = 3$, it can be deduced that D_3 will be generated by a RGSPP and a LGSPP. Recall that $|D_3| = 2 \times 3 = 6$, so $S_3 = D_3$. Thus Theorem 3.2 generalizes Lemma 3.2.

Rotations and Reflections Geometrically, in Theorem 3.2, σ is a rotation of the regular polygon P_n through an angle $\frac{2\pi}{n}$ in its own plane, and τ is a reflection (or a turning over) in the diameter through the vertex 1. It looks like a RGSPP and a LGSPP are formed by rotation and reflection respectively. But there is a contradiction in S_4 which can be traced from a subgroup of S_4 particularly the Klein four-group. The Klein four-group is the group of symmetries of a four sided non-regular polygon(rectangle). The elements are:

$$e = I = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix},$$

$$\delta_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix},$$

$$\delta_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix},$$

and

$$\delta_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

Observe the following:

$$N_\rho(\delta_1) = 12343412 = (12)(34)(34)(12),\ N_\lambda(\delta_1) = 12342143.$$

$$N_\rho(\delta_2) = 12342143,\ N_\lambda(\delta_2) = 12343412 = (12)(34)(34)(12).$$

$$N_\rho(\delta_3) = 12344321 = 123(44)321,\ N_\lambda(\delta_3) = 12341234 = (1234)(1234).$$

So, δ_1 is a RGSPP while δ_2 is a LGSPP and δ_3 is a GSPP. Geometrically, δ_1 is a rotation through an angle of π while δ_2 and δ_3 are reflections in the axes of symmetry parallel to the sides. Thus δ_3 which is a GSPP is both a reflection and a rotation, which is impossible. Therefore, the geometric meaning of a RGSPP and a LGSPP are not rotation and reflection respectively. It is difficult to really ascertain the geometric meaning of a RGSPP and a LGSPP if at all it exist.

How beautiful will it be if $GSPP_\rho(S_n), PP_\rho(S_n), GSPP_\lambda(S_n), PP_\lambda(S_n), GSPP(S_n)$ and $PP(S_n)$ form algebraic structures under the operation of map composition.

Theorem 3.3 Let S_n be a symmetric group of degree n. If $\sigma \in S_n$, then

1. $\sigma \in PP_\lambda(S_n) \Leftrightarrow \sigma^{-1} \in PP_\lambda(S_n)$.
2. $\sigma \in PP_\rho(S_n) \Leftrightarrow \sigma^{-1} \in PP_\rho(S_n)$.
3. $I \in PP_\lambda(S_n)$.

Proof 1. $\sigma \in PP_\lambda(S_n)$ implies

$$N_\lambda(\sigma) = 12\cdots n\sigma(n)\cdots\sigma(2)\sigma(1)$$

is a palindrome. Consequently,

$$\sigma(n) = n,\ \sigma(n-1) = n-1, \cdots, \sigma(2) = 2,\ \sigma(1) = 1.$$

So

$$N_\lambda(\sigma^{-1}) = \sigma(1)\sigma(2)\cdots\sigma(n)n\cdots 21 = 12\cdots nn\cdots 21 \Rightarrow \sigma^{-1} \in PP_\lambda(S_n).$$

The converse is similarly proved by carrying out the reverse of the procedure above.

2. $\sigma \in PP_\rho(S_n)$ implies

$$N_\rho(\sigma) = 12\cdots n\sigma(1)\cdots\sigma(n-1)\sigma(n)$$

is palindrome. Consequently,

$$\sigma(1) = n,\ \sigma(2) = n-1, \cdots, \sigma(n-1) = 2,\ \sigma(n) = 1.$$

So

$$N_\rho(\sigma^{-1}) = \sigma(1)\sigma(2)\cdots\sigma(n-1)\sigma(n)12\cdots n = n\cdots 2112\cdots n \Rightarrow \sigma^{-1} \in PP_\rho(S_n).$$

The converse is similarly proved by carrying out the reverse of the procedure above.

3.

$$I = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}.$$

$$N_\lambda(I) = 12\cdots nn\cdots 21 \Rightarrow I \in PP_\lambda(S_n).$$

Theorem 3.4 Let S_n be a symmetric group of degree n. If $\sigma \in S_n$, then

1.$\sigma \in GSPP_\lambda(S_n) \iff \sigma^{-1} \in GSPP_\lambda(S_n)$.

2.$\sigma \in GSPP_\rho(S_n) \iff \sigma^{-1} \in GSPP_\rho(S_n)$.

3.$I \in GSPP(S_n)$.

Proof If $\sigma \in S_n$, then

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}.$$

So

$$N_\lambda(\sigma) = 12\cdots n\sigma(n)\cdots\sigma(2)\sigma(1)$$

and

$$N_\rho(\sigma) = 12\cdots n\sigma(1)\cdots\sigma(n-1)\sigma(n)$$

are numbers with even number of digits whether n is an even or odd number. Thus, $N_\rho(\sigma)$ and $N_\lambda(\sigma)$ are GSPs defined by

$$a_1a_2a_3\cdots a_na_n\cdots a_3a_2a_1$$

and not

$$a_1a_2a_3\cdots a_{n-1}a_na_{n-1}\cdots a_3a_2a_1$$

where all $a_1, a_2, a_3, \cdots, a_n \in \mathbb{N}$ having one or more digits because the first has even number of digits (or grouped digits) while the second has odd number of digits (or grouped digits). The following grouping notations will be used:

$$(a_i)_{i=1}^n = a_1a_2a_3\cdots a_n$$

and

$$[a_i]_{i=1}^n = a_na_{n-1}a_{n-2}\cdots a_3a_2a_1.$$

Let $\sigma \in S_n$ such that

$$\sigma = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ \sigma(x_1) & \sigma(x_2) & \cdots & \sigma(x_n) \end{pmatrix}.$$

where $x_i \in \mathbb{N}, \forall\ i \in \mathbb{N}$.

1. So $\sigma \in GSPP_\lambda(S_n)$ implies

$$N_\lambda(\sigma) = (x_{i_1})_{i_1=1}^{n_1}(x_{i_2})_{i_2=(n_1+1)}^{n_2}(x_{i_3})_{i_3=(n_2+1)}^{n_3}\cdots(x_{i_{n-1}})_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}}(x_{i_n})_{i_n=(n_{n-1}+1)}^{n_n}$$

$$[\sigma(x_{i_n})]_{i_n=(n_{n-1}+1)}^{n_n}[\sigma(x_{i_{n-1}})]_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}}\cdots[\sigma(x_{i_3})]_{i_3=(n_2+1)}^{n_3}[\sigma(x_{i_2})]_{i_2=(n_1+1)}^{n_2}[\sigma(x_{i_1})]_{i_1=1}^{n_1}$$

is a GSP, where $x_{i_j} \in \mathbb{N}$, $\forall\ i_j \in \mathbb{N}$, $j \in \mathbb{N}$ and $n_n = n$. The interval of integers $[1, n]$ is partitioned into

$$[1, n] = [1, n_1] \cup [n_1 + 1, n_2] \cup \cdots \cup [n_{n-2} + 1, n_{n-1}] \cup [n_{n-1}, n_n].$$

The length of each grouping $(\cdot)_{i_j}^{n_j}$ or $[\cdot]_{i_j}^{n_j}$ is determined by the corresponding interval of integers $[n_i + 1, n_{i+1}]$ and it is a matter of choice in other to make the number $N_\lambda(\sigma)$ a GSP.

Now that $N_\lambda(\sigma)$ is a GSP, the following are true:

$$(x_{i_n})_{i_n=(n_{n-1}+1)}^{n_n} = [\sigma(x_{i_n})]_{i_n=(n_{n-1}+1)}^{n_n} \Leftrightarrow [x_{i_n}]_{i_n=(n_{n-1}+1)}^{n_n} = (\sigma(x_{i_n}))_{i_n=(n_{n-1}+1)}^{n_n}$$

$$(x_{i_{n-1}})_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}} = [\sigma(x_{i_{n-1}})]_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}} \Leftrightarrow [x_{i_{n-1}}]_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}} = (\sigma(x_{i_{n-1}}))_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$(x_{i_2})_{i_2=(n_1+1)}^{n_2} = [\sigma(x_{i_2})]_{i_2=(n_1+1)}^{n_2} \Leftrightarrow [x_{i_2}]_{i_2=(n_1+1)}^{n_2} = (\sigma(x_{i_2}))_{i_2=(n_1+1)}^{n_2}$$

$$(x_{i_1})_{i_1=1}^{n_1} = [\sigma(x_{i_1})]_{i_1=1}^{n_1} \Leftrightarrow [x_{i_1}]_{i_1=1}^{n_1} = (\sigma(x_{i_1}))_{i_1=1}^{n_1}$$

Therefore, since

$$\sigma = \begin{pmatrix} x_1 & \cdots & x_{i_1} & \cdots & x_{n_1} & \cdots & x_{n_{n-1}+1} & \cdots & x_{j_k} & \cdots & x_{n_n} \\ \sigma(x_1) & \cdots & \sigma(x_{i_1}) & \cdots & \sigma(x_{n_1}) & \cdots & \sigma(x_{n_{n-1}+1}) & \cdots & \sigma(x_{j_k}) & \cdots & \sigma(x_{n_n}) \end{pmatrix},$$

then

$$\sigma^{-1} = \begin{pmatrix} \sigma(x_1) & \cdots & \sigma(x_{i_1}) & \cdots & \sigma(x_{n_1}) & \cdots & \sigma(x_{n_{n-1}+1}) & \cdots & \sigma(x_{j_k}) & \cdots & \sigma(x_{n_n}) \\ x_1 & \cdots & x_{i_1} & \cdots & x_{n_1} & \cdots & x_{n_{n-1}+1} & \cdots & x_{j_k} & \cdots & x_{n_n} \end{pmatrix},$$

so

$$N_\lambda(\sigma^{-1}) = (\sigma(x_{i_1}))_{i_1=1}^{n_1}(\sigma(x_{i_2}))_{i_2=(n_1+1)}^{n_2}(\sigma(x_{i_3}))_{i_3=(n_2+1)}^{n_3}\cdots(\sigma(x_{i_{n-1}}))_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}}$$

$$(\sigma(x_{i_n}))_{i_n=(n_{n-1}+1)}^{n_n}[x_{i_n}]_{i_n=(n_{n-1}+1)}^{n_n}[x_{i_{n-1}}]_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}}\cdots[x_{i_3}]_{i_3=(n_2+1)}^{n_3}[x_{i_2}]_{i_2=(n_1+1)}^{n_2}[x_{i_1}]_{i_1=1}^{n_1}$$

is a GSP. Hence, $\sigma^{-1} \in GSPP_\lambda(S_n)$.

The converse can be proved in a similar way since $(\sigma^{-1})^{-1} = \sigma$.

2. Also, $\sigma \in GSPP_\rho(S_n)$ implies

$$N_\rho(\sigma) = (x_{i_1})_{i_1=1}^{n_1}(x_{i_2})_{i_2=(n_1+1)}^{n_2}(x_{i_3})_{i_3=(n_2+1)}^{n_3}\cdots(x_{i_{n-1}})_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}}(x_{i_n})_{i_n=(n_{n-1}+1)}^{n_n}$$

$$(\sigma(x_{i_1}))_{i_1=1}^{n_1}(\sigma(x_{i_2}))_{i_2=(n_1+1)}^{n_2}(\sigma(x_{i_3}))_{i_3=(n_2+1)}^{n_3}\cdots(\sigma(x_{i_{n-1}}))_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}}(\sigma(x_{i_n}))_{i_n=(n_{n-1}+1)}^{n_n}$$

is a GSP, where $x_{i_j} \in \mathbb{N}$, $\forall\ i_j \in \mathbb{N}$, $j \in \mathbb{N}$ and $n_n = n$. The interval $[1, n]$ is partitioned into

$$[1, n] = [1, n_1] \cup [n_1 + 1, n_2] \cup \cdots \cup [n_{n-2} + 1, n_{n-1}] \cup [n_{n-1}, n_n].$$

The length of each grouping $(\cdot)_{i_j}^{n_j}$ is determined by the corresponding interval of integers $[n_i + 1, n_{i+1}]$ and it is a matter of choice in other to make the number $N_\rho(\sigma)$ a GSP.

Now that $N_\rho(\sigma)$ is a GSP, the following are true:

$$(x_{i_n})_{i_n=(n_{n-1}+1)}^{n_n} = (\sigma(x_{i_1}))_{i_1=1}^{n_1}$$

$$(x_{i_{n-1}})_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}} = (\sigma(x_{i_2}))_{i_2=(n_1+1)}^{n_2}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$(x_{i_2})_{i_2=(n_1+1)}^{n_2} = (\sigma(x_{i_{n-1}}))_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}}$$

$$(x_{i_1})_{i_1=1}^{n_1} = (\sigma(x_{i_n}))_{i_n=(n_{n-1}+1)}^{n_n}.$$

Therefore, since

$$\sigma = \begin{pmatrix} x_1 & \cdots & x_{i_1} & \cdots & x_{n_1} & \cdots & x_{n_{n-1}+1} & \cdots & x_{j_k} & \cdots & x_{n_n} \\ \sigma(x_1) & \cdots & \sigma(x_{i_1}) & \cdots & \sigma(x_{n_1}) & \cdots & \sigma(x_{n_{n-1}+1}) & \cdots & \sigma(x_{j_k}) & \cdots & \sigma(x_{n_n}) \end{pmatrix},$$

then

$$\sigma^{-1} = \begin{pmatrix} \sigma(x_1) & \cdots & \sigma(x_{i_1}) & \cdots & \sigma(x_{n_1}) & \cdots & \sigma(x_{n_{n-1}+1}) & \cdots & \sigma(x_{j_k}) & \cdots & \sigma(x_{n_n}) \\ x_1 & \cdots & x_{i_1} & \cdots & x_{n_1} & \cdots & x_{n_{n-1}+1} & \cdots & x_{j_k} & \cdots & x_{n_n} \end{pmatrix}.$$

So

$$N_\rho(\sigma^{-1}) = (\sigma(x_{i_1}))_{i_1=1}^{n_1}(\sigma(x_{i_2}))_{i_2=(n_1+1)}^{n_2}(\sigma(x_{i_3}))_{i_3=(n_2+1)}^{n_3}\cdots(\sigma(x_{i_{n-1}}))_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}}$$

$$(\sigma(x_{i_n}))_{i_n=(n_{n-1}+1)}^{n_n}(x_{i_1})_{i_1=1}^{n_1}(x_{i_2})_{i_2=(n_1+1)}^{n_2}(x_{i_3})_{i_3=(n_2+1)}^{n_3}\cdots(x_{i_{n-1}})_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}}(x_{i_n})_{i_n=(n_{n-1}+1)}^{n_n}$$

is a GSP. Hence, $\sigma^{-1} \in GSPP_\rho(S_n)$.

The converse can be proved in a similar way since $(\sigma^{-1})^{-1} = \sigma$.

3.

$$I = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}.$$

$N_\lambda(I) = 12\cdots nn\cdots 21 = 12\cdots(nn)\cdots 21 \Longrightarrow I \in GSPP_\lambda(S_n)$ and

$$N_\rho(I) = (12\cdots n)(12\cdots n) \Longrightarrow I \in GSPP_\rho(S_n)$$

thus $I \in GSPP(S_n)$.

§4. Conclusion and Future studies

By Theorem 3.1, it is certainly true in every symmetric group S_n of degree n there exist at least a RGSPP and a LGSPP(although they are actually RPP and LPP). Following Example 3.1, there are 2 RGSPPs, 2 LGSPPs and 2 GSPPs in S_2 while from Lemma 3.1, there are 4 RGSPPs, 4 LGSPPs and 2 GSPPs in S_3. Also, it can be observed that

$$|GSPP_\rho(S_2)| + |GSPP_\lambda(S_2)| - |GSPP(S_2)| = 2! = |S_2|$$

and

$$|GSPP_\rho(S_3)| + |GSPP_\lambda(S_3)| - |GSPP(S_3)| = 3! = |S_3|.$$

The following problems are open for further studies.

Problem 4.1 1. How many RGSPPs, LGSPPs and GSPPs are in S_n?
2. Does there exist functions f_1, f_2, $f_3 : \mathbb{N} \to \mathbb{N}$, such that $|GSPP_\rho(S_n)| = f_1(n)$, $|GSPP_\lambda(S_n)| = f_2(n)$ and $|GSPP(S_n)| = f_3(n)$?
3. In general, does the formula

$$|GSPP_\rho(S_n)| + |GSPP_\lambda(S_n)| - |GSPP(S_n)| = n! = |S_n|?$$

hold. If not, for what other $n > 3$ is it true?

The GAP package or any other appropriate mathematical package could be helpful in investigating the solutions to them.

If the first question is answered, then the number of palindromes that can be formed from the set $\{1, 2, \cdots, n\}$ can be known since in the elements of S_n, the bottom row gives all possible permutation of the integers $1, 2, \cdots, n$.

The Cayley Theorem(Theorem 2.1) can also be used to make a further study on generalized Smarandache palindromic permutations. In this work, $\mathbb{N}$ was the focus and it does not contain the integer zero. This weakness can be strengthened by considering the set $\mathbb{Z}_n = \{0, 1, 2, \cdots, n-1\}, \forall\ n \in \mathbb{N}$. Recall that $(\mathbb{Z}_n, +)$ is a group and so by Theorem 2.1 $(\mathbb{Z}_n, +)$ is isomorphic to a permutation group particularly, one can consider a subgroup of the symmetric group $S_{\mathbb{Z}_n}$.

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On certain inequalities involving the Smarandache function

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Abstract Let $S(n)$ be the Smarandache function defined by $S(n) = \min\{m \geq 1 : \ n|m!\}$. The aim of this note is to prove that for each $k \geq 2$ there are infinitely many positive integers $m_1, \ldots, m_k$ and $n_1, \ldots, n_k$ such that $S(m_1 + m_2 + \cdots + m_k) < S(m_1) + S(m_2) + \cdots + S(m_k)$, and $S(n_1 + n_2 + \cdots + n_k) > S(n_1) + S(n_2) + \cdots + S(n_k)$.

Keywords Vinogradov's three-primes theorem, inequalities.

§1. Introduction

For any positive integer n, the famous Smarandache function is defined by

$$S(n) = \min\{m \geq 1 : \ n|m!\}$$

For many properties, problems, generalizations and extensions of this function see e.g. Smarandache's well-known book [5], or the author's book [3].

Recently, Lu Yaming [6] has proved that for each positive integer k, the equation

$$S(x_1 + x_2 + \cdots + x_k) = S(x_1) + S(x_2) + \cdots + S(x_k)$$

has infinitely many positive integer solutions $x_1, \ldots, x_k$. In what follows, we shall prove that both of inequalities $S(a + b) < S(a) + S(b)$ and $S(c + d) > S(c) + S(d)$ have infinitely many positive integer solutions a, b, c, d; and more generally the same is true for $S(m_1 + \cdots + m_k) < S(m_1) + \cdots + S(m_k)$, and $S(n_1 + \cdots + n_k) > S(n_1) + \cdots + S(n_k)$, for each $k \geq 2$.

§2. Proof of the theorem

The fact that $S(a + b) < S(a) + S(b)$ has infinitely many solutions, follows from well known inequalities. For example, $a = n!$ and $b = (n + 1)!$ are solutions. Indeed, as $a + b = n! + (n + 1)! = n!(n + 2)$, and $n!(n + 2)$ dividing $(n + 2)!$, one has clearly $S(a + b) \leq n + 2 < 2n + 1 = S(n!) + S((n+1)!)$, as $S(n!) = n$, etc. For another solutions, put $a = p! + 1$, $b = p! - 1$, where $p \geq 2$ is a prime. Then, it is known that (see e.g. [2], [3]) $S(p! + 1) > p$, $S(p! - 1) > p$, so we get $S(p! + 1) + S(p! - 1) > 2p \geq S(2p!)$, as $S(2p!) \leq 2S(p!) = 2p$, by the inequality $S(mn) \leq nS(m)$, see [1], [3].

Let now c, d be positive integers such that $c+d$ is a prime, and $c > 4$, $d > 4$. Then c and d cannot be both primes, since then $c+d$ would be even (> 4). Then $S(c+d) = c+d > S(c)+S(d)$, by the well-known fact (see e.g. [3], [5]) that $S(n) \leq n$, with equality only for $n = 4$, and $n =$ prime.

The last part of the above proof may be easily extended to the case of k numbers. Namely, let $n_1, n_2, \ldots, n_k$ be positive integers, not all of which are primes, so that $n_1 + n_2 + \cdots + n_k$ is a prime. (It is easy to see the possibility of constructing such numbers). Then $S(n_1 + \cdots + n_k) = n_1 + \cdots + n_k > S(n_1) + \cdots + S(n_k)$, as above.

For the extension of the first part to arbitrary k, however, we need another argument. As in [6], we need Vinogradov's three-primes theorem, as stated in Lemma 1, and its generalization, as stated in Lemma 2:

Lemma 1. There exists a constant $K > 0$, such that each odd integer $n > K$ can be written as $n = p_1 + p_2 + p_3$, where $p_i (i = \overline{1,3})$ are odd primes.

Lemma 2. If $k \geq 3$ is an odd integer, then any sufficiently large odd integer n can be written as a sum of k odd primes

$$n = p_1 + p_2 + \cdots + p_k.$$

This follows immediately from Lemma 1, by induction see [6].

Let now $k \geq 3$ be odd. Then the odd integer $n = 4^s - 1$ for sufficiently large $s \geq s_0$ may be written as

$$4^s - 1 = p_1 + \cdots + p_k.$$

On the other hand, $S(4^s - 1) < 4^s - 1$, as $4^s - 1 \equiv 0 \pmod 3$, so $4^s - 1 \neq$ prime, $\neq 4$. This implies

$$S(p_1 + \cdots + p_k) = S(4^s - 1) < p_1 + \cdots + p_k = S(p_1) + \cdots + S(p_k).$$

If $k \geq 4$ is even, then for sufficiently large s, $3^s - 2$ may be written as

$$3^s - 2 = p_1 + \cdots + p_{k-1}$$

implying

$$3^s = 2 + p_1 + \cdots + p_{k-1},$$

i.e. $S(3^s) = S(2 + p_1 + \cdots + p_{k-1}) < 3^s = 2 + p_1 + \cdots + p_{k_1} = S(2) + S(p_1) + \cdots + S(p_{k-1})$, 3^s not being a prime.

This finishes the proof of the theorem.

Remark. The arithmetical function occurring in the first part of the proof, namely

$$F(n) = \min\{m \geq 0 : \ n | m!(m+2)\}$$

(where $0! = 1$) has a close analogy with the Smarandache function. We have called it in [4], as the modification of the Smarandache function. In [4] we have proved that $F(p) = p - 1$ for all primes p; $F(k!) = k + 1$ for $k \geq 3$; $F(1!) = F(2!) = 1$; and that $S(n) < F(n)$ or $S(m) > F(m)$ both have infinitely many solutions n, m.

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Sequences of Pentagonal numbers[1]

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Abstract Shyam Sunder Gupta [3] has defined Smarandache consecutive and reversed Smarandache sequences of triangular numbers. Delfim F.M.Torres and Viorica Teca [1] have further investigated these sequences and defined mirror and symmetric Smarandache sequences of triangular numbers making use of Maple system. Working on the same lines we have defined and investigated consecutive, reversed, mirror and symmetric Smarandache sequences of pentagonal numbers of dimension 2 using the Maple system .

Keywords Figurate number, Smarandache consecutive sequence, Smarandache mirror sequence.

§1. Introduction

Figurate number is a number which can be represented by a regular geometrical arrangement of equally spaced points. If the arrangement forms a regular polygon the number is called a polygonal number. Different figurate sequences are formed depending upon the dimension we consider. Each dimension gives rise to a system of figurate sequences which are infinite in number.

In this paper we consider a figurate sequence of pentagonal numbers of dimension 2. Henceforth, unless and otherwise stated, "pentagonal numbers" will mean pentagonal numbers of dimension 2.

The n-th pentagonal number $t_n, n \in N$ of dimension 2 is defined by $t_n = 2n+\frac{5}{2}n(n-1)-n^2$.

We can obtain the first k terms of Pentagonal numbers in Maple as:

```
> t:= n->2*n + (5\2)*n*(n-1)-n^{2}:

> first := k -> seq(t(n), n=1...20);
```

For example first 20 terms are:

```
> first(20);

1, 5, 12, 22, 35, 51, 70, 92, 117, 145, 176, 210, 247, 287, 330,
      376, 425, 477, 532, 590
```

For constructing Smarandache sequence of pentagonal numbers we require the operation of concatenation on the terms of above sequence which is defined in Maple as;

[1]This work is supported by UGC under Minor project F. No. 23-245/06.

```
> conc :=(n,m) -> n*10^length(m)+m:
```

We define Smarandache consecutive sequence $\{scs_n\}$ for pentagonal numbers recursively as;
$scs_1 = u_1$,
$scs_n = conc(scs_{n-1}, u_n)$
Using Maple We have obtained first 20 terms of Smarandache consecutive sequence of pentagonal numbers;

```
> conc :=(n,m)-> n*10^length(m)+m:
> scs_n := (u,n)-> if n = 1 then u(1)else conc(scs_n(u,n-1),u(n)) fi:
> scs := (u,n)-> seq (scs_n(u,i),i=1...n):
> scs(t,20);

  1, 15, 1512, 151222, 15122235, 1512223551, 151222355170,
        15122235517092, 151222355170921 17, 1512223551709211 7145,
        1512223551709211 7145176, 1512223551709211 7145176210,
        1512223551709211 7145176210247,
        1512223551709211 71451762102472 87,
        1512223551709211 714517621024728 7330,
        1512223551709211 7145176210247287330376,
        1512223551709211 71451762102472873303764 25,
        1512223551709211 714517621024728733037642 5477,
        1512223551709211 7145176210247287330376425477532,
        1512223551709211 71451762102472873303764254775325 90
```

Same sequence can be displayed in "triangular form" as;

```
> show := L -> map(i ->print(i),L):
> show([scs(t,20)]);

                                    1
                                   15
                                 1512
                               151222
                             15122235
                           1512223551
                         151222355170
                       15122235517092
                     151222355170921 17
                   1512223551709211 7145
                 1512223551709211 7145176
               1512223551709211 7145176210
             1512223551709211 71451762102 47
           1512223551709211 714517621024728 7
         1512223551709211 7145176210247287330 4
       1512223551709211 71451762102472873303764 25
     1512223551709211 714517621024728733037642 5477
   1512223551709211 7145176210247287330376425477532
 1512223551709211 71451762102472873303764254775325 90
```

```
151222355170921171451762102472873 30
151222355170921171451762102472873 3037 6
151222355170921171451762102472873 303763 425
151222355170921171451762102472873 30376425 477
151222355170921171451762102472873 303764254 77532
151222355170921171451762102472873 3037642547 7532590
```

The reversed Smarandache sequence (rss)associated with a given sequence $\{u_n\}, n \in N$ is defined recursively as;

$$rss_1 = u_1,$$
$$rss_n = conc(u_n, rss_{n-1}).$$

In Maple we use the following program;

```
> rss_n :=(u,n) -> if n=1 then u(1) else conc(u(n),rss_n(u,n-1)) fi:
> rss := (u,n) -> seq(rss_n(u,i),i=1..n):
```

We get the first 20 terms of reversed Smarandache sequence of pentagonal numbers;

```
> rss(t,20);
```

```
  1, 51, 1251, 221251, 35221251, 5135221251, 705135221251,
       92705135221251, 11792705135221251, 1451179270513522 1251,
       176145117927051352212 51, 21017614511792705135221251,
       247210176145117927051352212 51,
       2872472101761451179270513522 1251,
       33028724721017614511792705135221251,
       376330287247210176145117927051352212 51,
       4253763302872472101761451179270513522 1251,
       47742537633028724721017614511792705135221251,
       532477425376330287247210176145117927051352212 51,
       5905324774253763302872472101761451179270513522 1251
```

Smarandache Mirror Sequence (sms)is defined as follows:

$$sms_1 = u_1,$$
$$sms_n = conc(conc(u_n, sms_{n-1}), u_n).$$

The following program gives first 20 terms of Smarandache Mirror sequence of pentagonal numbers.

```
> sms_n := (u,n) -> if n=1 then
```

```
> u(1)
> else
> conc(conc(u(n),sms_n(u,n-1)),u(n))
> fi:
> sms :=(u,n) ->seq(sms_n(u,i), i=1..n):
> sms(t,20);

  1, 515, 1251512, 22125151222, 352212515122235, 5135221251512223551,
        7051352212515122235517 0, 92705135221251512223551709 2,
        1179270513522125151222355170921 17,
        14511792705135221251512223551709211 7145,
        176145117927051352212515122235517092117145176,
        2101761451179270513522125151222355170921171451762 10,
        24721017614511792705135221251512223551709211714517621 0247,
        28724721017614511792705135221251512223551709211714517621 02\
        47287, 330287247210176145117927051352212515122235517092117 1\
        45176210247287330, 37633028724721017614511792705135221251 51\
        22235517092117145176210247287330376, 42537633028724721017 61\
        45117927051352212515122235517092117145176210247287330376 42\
        5, 47742537633028724721017614511792705135221251512223551709\
        21171451762102472873303764254 77, 53247742537633028724721017\
        61451179270513522125151222355170921171451762102472873303764\
        25477532, 59053247742537633028724721017614511792705135221 2\
        51512223551709211714517621024728733037642 5.
```

Finally Smarandache Symmetric sequence (sss) is defined as:

$$sss_{2n-1} = conc(bld(scs_{2n-1}), rss_{2n-1}),$$

$$sss_{2n} = conc(scs_{2n}, rss_{2n}), n \in N.$$

where the function "bld" (But Last Digit) is defined in Maple as:

```
> bld := n->iquo(n,10):
```

First 20 terms of Smarandache Symmetric sequence are obtained in Maple as:

```
> bld :=n-> iquo(n,10):
> conc := (n,m)-> n*10^length(m)+m:
> sss_n := (u,n) -> if type(n,odd) then
> conc(bld(scs_n(u,(n+1)/2)),rss_n(u,(n+1)/2))
> else
> conc(scs_n(u,n/2),rss_n(u,n/2))
> fi:
> sss := (u,n) -> seq(sss_n(u,i), i=1..n):
> sss(t,20);
```

```
1, 11, 151, 1551, 1511251, 15121251, 15122221251, 151222221251,
      151222335221251, 1512223535221251, 1512223555135221251,
      15122235515135221251, 151222355177051352221251,
      1512223551707051352221251, 15122235517099270513522125 1,
      151222355170929270513522125 1,
      15122235517092111179270513522125 1,
      15122235517092117117927051352212 51,
      151222355170921171414511792705135221251,
      1512223551709211714514511792705135221251.
```

We find out primes from a large number (first 500 terms) of various Smarandache sequences defined so far. We have used Maple 6 on Pentium 3 with 256 Mb RAM. We first collect the lists of first 500 terms of the consecutive, reversed, mirror and symmetric sequences of Pentagonal numbers:

```
 > st :=time(): Lscs500:=[scs(t,500)]: printf("%a seconds",round(time()-st));
16 seconds
> st :=time(): Lrss500:=[rss(t,500)]: printf("%a seconds",round(time()-st));

18 seconds
> st :=time(): Lsms500:=[sms(t,500)]: printf("%a seconds",round(time()-st));
50 seconds
> st :=time(): Lsss500:=[sss(t,500)]: printf("%a seconds",round(time()-st));
11 seconds
```

Further we find the number of digits in the 500th term of each sequence:

```
> length(Lscs500[500]),length(Lrss500[500]);

                              2626, 2626

> length(Lsms500[500]),length(Lsss500[500]);

                              5251, 2268
```

There exist no prime in the first 500 terms of Smarandache consecutive sequence of pentagonal numbers.

```
>st:= time():select(isprime,Lscs500);
> printf("%a minutes",round((time()-st)/60));

                                  [ ]
```

```
11 minutes
```

There are only two primes in the first 500 terms of reversed Smarandache sequence of pentagonal numbers.

```
>st:= time():select(isprime,Lrss500);
[221251,
100492997179894598176974109664795887951309437693625928779213291390
906518991589182884528772587001862808556284847841358342682720820 1
781317806207992679235785477786277180765017582575152744827381573 15
172490718327117770525698766923068587679476731066676660456541764792
641706355162935623226171261105605015990059302587075811557526569405
635755777552005462654055534875292252360518015124550692501424959549
051485104797247437469054637645850453274480744290437764326542757422
524175041251407554026239772392853880138320378423736736895364263596
035497350373458034126336753322732782323403190131465310323060230175 2
975129330289122849728085276762727026867264672607025676252852489724 5
122413023751233752300222632222652190121540211822082720475201261978 0
194371909718760184261809517767174421712016801164851617215862155551 5
251149501465214357140651377613490132071292712650123761210511837115 7
211310110511079510542102921004598019560932290878855862684008177795 7
774075267315710769026700650163056112592257355551537051925017484546 7
645104347418740303876372535773432329031513015288227522625250123802 2
622147203519261820171716171520142613351247116210801001925852782715 6
515905324774253763302872472101761451179270513522125 1]
```

```
> printf("%a minutes",round((time()-st)/60));
45 minutes
```

There is no prime in the first 500 terms of Smarandache mirror sequence.

```
>st:= time():select(isprime,Lsms500);
> printf("%a minutes",round((time()-st)/60));
```

[ ]

```
312 minutes
```

There are 4 primes in first 500 terms of Smarandache symmetric sequence of pentagonal numbers.

```
> st:= time():
```

```
> select(isprime,Lsss500);
> printf("%a minutes",round((time()-st)/60));

  [11, 151, 151222355170921171451451179270513522125 1, 1512223551709\
        211714517621024728733037642547753259065171578285292510010 0\
        192585278271565159053247742537633028724721017614511792705 1\
        35221251]

50 minutes
```

There are some results which can be obtained by the readers as; How many pentagonal numbers are there in Smarandache consecutive , mirror and symmetric sequences of pentagonal numbers ?

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On the additive analogues of the simple function

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Abstract The main purpose of this paper is using the analytic method to study the convergent properties of the Jozse Sandor's function, and obtain some interesting results.

Keywords Riemann zeta-function, convergent, asymptotic formula.

§1. Introduction and conclusions

For any positive integer n, the function $Z_*(n)$ and $Z(n)$ will be defined as

$$Z_*(n) = \max\left\{m \in N : \frac{m(m+1)}{2} \leq n\right\},$$

and

$$Z(n) = \min\left\{m \in N : n \leq \frac{m(m+1)}{2}\right\}.$$

That is, $Z_*(n)$ denotes the greatest positive integer m such that $\frac{m(m+1)}{2} \leq n$, and $Z(n)$ denotes the smallest positive integer m such that $n \leq \frac{m(m+1)}{2}$.

For example: $Z_*(1) = 1$, $Z_*(2) = 1$, $Z_*(3) = 2$, $Z_*(4) = 2$, $Z_*(5) = 2$, $Z_*(6) = 3$, $Z_*(7) = 3$, $Z_*(8) = 3$, $Z_*(9) = 3, \cdots$ and $Z(1) = 1$, $Z(2) = 2$, $Z(3) = 2$, $Z(4) = 3$, $Z(5) = 3$, $Z(6) = 3$, $Z(7) = 4$, $Z(8) = 4$, $Z(9) = 4$, $Z(10) = 4, \cdots$.

Recently, Jozsef Sandor had studied the properties of these functions, and proved some interesting conclusions, one of them is

$$\sum_{n=1}^{\infty} \frac{1}{(Z_*(n))^s}$$

is convergent for $s > 2$, and divergent for $s \leq 2$.

In this paper, we shall use the elementary methods to study the convergent properties of some series involving the Jozsef Sandor's functions, and obtain some interesting identities. That is, we shall prove the following several theorems:

Theorem 1. For any complex number s, the infinite series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(Z_*(n))^s},$$

and

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{(Z(n))^s},$$

are convergent if $s>0$, and divergent if $s\leq 0$.

Theorem 2. For any complex number s with Re $s>2$, we have the identities

$$\sum_{n=1}^{\infty}\frac{1}{(Z_*(n))^s}=\zeta(s-1)+\zeta(s),$$

and

$$\sum_{n=1}^{\infty}\frac{1}{(Z(n))^s}=\zeta(s-1),$$

where $\zeta(s)$ denotes the Riemann zeta-function.

Theorem 3. For any positive integer n and complex number s with Re $s>1$, we have

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{(Z_*(n))^s}=\frac{2}{4^s}\zeta(s)-\frac{1}{2^s}\zeta(s),$$

and

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{(Z(n))^s}=\left(1-\frac{1}{2^s}\right)\zeta(s)-\frac{2}{4^s}\zeta\left(s,\frac{1}{4}\right),$$

where $\zeta(s,\alpha)$ denotes the Hurwitz zeta-function.

From Theorem 3 we may immediately deduce the following:

Corollary. Let $Z_*(n)$ defined as the above, then we have

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{(Z_*(n))^2}=-\frac{\pi^2}{16}.$$

§2. Proof of the theorems

In this section, we shall complete the proof of Theorems.

First we prove Theorem 1.

If

$$\frac{m(m+1)}{2}\leq n<\frac{(m+1)(m+2)}{2},$$

then $Z_*(n)=m$ repeated

$$\frac{(m+1)(m+2)}{2}-\frac{m(m+1)}{2}=m+1$$

times, so we have

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{(Z_*(n))^s}=\sum_{\substack{n=1\\ Z_*(n)=m}}^{\infty}\frac{(-1)^n(m+1)}{m^s}$$

If m is an odd, every term is concealed by positive and negative addition.

If m is an even, only one term will be retained in every repeated term, so we can obtain

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{(Z_*(n))^s}=-\frac{1}{2^s}+\frac{1}{4^s}-\frac{1}{6^s}+\cdots+\frac{(-1)^n}{(2n)^s}+\cdots=-\frac{1}{2^s}\left(\frac{1}{1^s}-\frac{1}{2^s}+\frac{1}{3^s}+\cdots\right).$$

Using the same method, we can also obtain

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{(Z(n))^s}=-\frac{1}{1^s}+\frac{1}{3^s}-\frac{1}{5^s}+\frac{1}{7^s}+\cdots+\frac{(-1)^n}{(2n-1)^s}+\cdots.$$

Now Theorem 1 follows from the Leibniz convergent criterion.

If

$$\frac{m(m+1)}{2}\leq n<\frac{(m+1)(m+2)}{2},$$

then $Z_*(n)$ repeated

$$\frac{(m+1)(m+2)}{2}-\frac{m(m+1)}{2}=m+1$$

times.

Hence, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{(Z_*(n))^s} &= \sum_{m=1}^{\infty}\frac{m+1}{m^s}\\ &= \zeta(s-1)+\zeta(s).\end{aligned}$$

Using the same method, we can also obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{(Z(n))^s} &= \sum_{m=1}^{\infty}\frac{\frac{m(m+1)}{2}-\frac{(m-1)m}{2}}{m^s}\\ &= \zeta(s-1).\end{aligned}$$

This completes the proof of Theorem 2.

Now we come to prove Theorem 3. From the method of proving Theorem 1 and Theorem 2, we can easily get

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n}{(Z_*(n))^s} &= -\frac{1}{2^s}+\frac{1}{4^s}-\frac{1}{6^s}+\cdots+\frac{(-1)^n}{(2n)^s}+\cdots\\ &= -\left(\frac{1}{2^s}+\frac{1}{6^s}+\frac{1}{10^s}+\cdots+\frac{1}{(4n-2)^s}+\cdots\right)\\ &\quad+\left(\frac{1}{4^s}+\frac{1}{8^s}+\cdots+\frac{1}{(4n)^s}+\cdots\right)\\ &= -\frac{1}{2^s}\left(\frac{1}{1^s}+\frac{1}{3^s}+\cdots+\frac{1}{(2n-1)^s}+\cdots\right)\\ &\quad+\frac{1}{2^s}\left(\frac{1}{2^s}+\frac{1}{4^s}+\cdots+\frac{1}{(2n)^s}+\cdots\right)\\ &= \frac{2}{4^s}\zeta(s)-\frac{1}{2^s}\zeta(s),\end{aligned}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n}{(Z(n))^s} &= -\frac{1}{1^s}+\frac{1}{3^s}-\frac{1}{5^s}+\frac{1}{7^s}+\cdots+\frac{(-1)^n}{(2n-1)^s}+\cdots\\ &= \left(\frac{1}{1^s}+\frac{1}{3^s}+\frac{1}{5^s}+\cdots+\frac{1}{(2n-1)^s}+\cdots\right)\\ &\quad -2\left(\frac{1}{1^s}+\frac{1}{5^s}+\cdots+\frac{1}{(4n-3)^s}+\cdots\right)\\ &= \left(1-\frac{1}{2^s}\right)\zeta(s)-\frac{2}{4^s}\zeta\left(s,\frac{1}{4}\right),\end{aligned}$$

where we have used the identities

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^s}=\left(1-\frac{1}{2^s}\right)\zeta(s),$$

and

$$\frac{1}{1^s}+\frac{1}{5^s}+\cdots+\frac{1}{(4n-3)^s}+\cdots=\sum_{n=0}^{\infty}\frac{1}{(4n+1)^s}=\frac{1}{4^s}\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{1}{4}\right)^s}=\frac{1}{4^s}\zeta\left(s,\frac{1}{4}\right).$$

This completes the proof of Theorem 3.

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On the primitive numbers of power p

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Abstract For any positive integer n and prime p, let $S_p(n)$ denotes the smallest positive integer m such that $m!$ is divisible by p^n. The main purpose of this paper is using the elementary method to study the properties of the summation $\sum_{d|n} S_p(d)$, and give an exact calculating formula for it.

Keywords Primitive number of power p, summation, calculating formula.

§1. Introduction and Results

Let p be a prime and n be any positive integer. Then we define the primitive number function $S_p(n)$ of power p as the smallest positive integer m such that $m!$ is divisible by p^n. For example, $S_3(1) = 3,\ S_3(2) = 6,\ S_3(3) = S_3(4) = 9,\ \cdots\cdots$. In problem 49 of book [1], Professor F.Smarandache asked us to study the properties of the sequence $\{S_p(n)\}$. About this problem, Zhang Wenpeng and Liu Duansen [3] had studied the asymptotic properties of $S_p(n)$, and obtained an interesting asymptotic formula for it. That is, for any fixed prime p and any positive integer n, they proved that

$$S_p(n) = (p-1)n + O\left(\frac{p}{\ln p}\ln n\right).$$

Yi Yuan [4] had studied the asymptotic property of $S_p(n)$ in the form

$$\frac{1}{p}\sum_{n\le x}|S_p(n+1) - S_p(n)|,$$

and obtained the following conclusion: For any real number $x \ge 2$, we have

$$\frac{1}{p}\sum_{n\le x}|S_p(n+1) - S_p(n)| = x\left(1-\frac{1}{p}\right) + O\left(\frac{\ln x}{\ln p}\right).$$

Xu Zhefeng [5] had studied the relationship between the Riemann zeta-function and an infinite series involving $S_p(n)$, and obtained some interesting identities and asymptotic formulae for $S_p(n)$. That is, for any prime p and complex number s with $\mathrm{Re}s > 1$, we have the identity:

$$\sum_{n=1}^{\infty}\frac{1}{S_p^s(n)} = \frac{\zeta(s)}{p^s - 1},$$

where $\zeta(s)$ is the Riemann zeta-function.

And, let p be a fixed prime, then for any real number $x \geq 1$,

$$\sum_{\substack{n=1 \\ S_p(n) \leq x}}^{\infty} \frac{1}{S_p(n)} = \frac{1}{p-1}\left(\ln x + \gamma + \frac{p \ln p}{p-1}\right) + O(x^{-\frac{1}{2}+\varepsilon}),$$

where γ is the Euler constant, ε denotes any fixed positive number.

Chen Guohui [7] had studied the calculating problem of the special value of the Smarandache function $S(n) = \min\{m : \ m \in N, \ n|m!\}$. That is, let p be a prime and k an integer with $1 \leq k < p$. Then for polynomials $f(x) = x^{n_k} + x^{n_{k-1}} + \cdots + x^{n_1}$ with $n_k > n_{k-1} > \cdots > n_1 \geq 1$, we have:

$$S(p^{f(p)}) = (p-1)f(p) + pf(1).$$

And, let p be a prime and k an integer with $1 \leq k < p$. Then for any positive integer n, we have:

$$S\left(p^{kp^n}\right) = k\left(\phi(p^n) + \frac{1}{k}\right)p,$$

where $\phi(n)$ is the Euler function. All these conclusions also hold for primitive function $S_p(n)$ of power p.

In this paper, we shall use the elementary method to study the calculating problem of the summation

$$\sum_{d|n} S_p(d),$$

and give some interesting calculating formulas for it. That is, we shall prove the following conclusions:

Theorem. For any prime p, we have the calculating formulas

(1) $\sum_{d|n} S_p(d) = p\sigma(n)$, if $1 \leq n \leq p$;

(2) $\sum_{d|n} S_p(d) = p\sigma(n) - (n-1)p$, if $p < n \leq 2p$, where $\sigma(n)$ denotes the summation over all divisors of n.

For general positive integer $n > 2p$, whether there exists a calculating formula for $\sum_{d|n} S_p(d)$ is an open problem.

§2. Proof of Theorem

To complete the proof of the theorem, we need a simple lemma which stated as following:

Lemma. For any prime p, we have:

(1) $S_p(d) = dp$, if $1 \leq d \leq p$;

(2) $S_p(d) = (d-k+1)p$, if $(k-1)p + k - 2 < d \leq kp$.

Proof. First we prove the case (1). From the definition of $S_p(n) = \min\{m : p^n|m!\}$ we know that to prove the case (1) of Lemma, we only to prove that $p^d \| (dp)!$. That is, $p^d|(dp)!$ and

$p^{d+1} \nmid (dp)!$. According to Theorem 1.7.2 of reference [6], we only to prove that $\sum_{j=1}^{\infty} \left[\frac{dp}{p^j}\right] = d$. In fact, if $1 \leq d < p$, note that $\left[\frac{d}{p^j}\right] = 0$ $(j = 1,\ 2,\ \cdots\cdots)$, we have

$$\sum_{j=1}^{\infty} \left[\frac{dp}{p^j}\right] = d + \left[\frac{d}{p}\right] + \left[\frac{d}{p^2}\right] + \cdots = d.$$

That is means $S_p(d) = dp$. If $d = p$, then $\sum_{d|n} \left[\frac{dp}{p^j}\right] = d + 1$, but $p^p \nmid (p^2 - 1)!$ and $p^p | p^2!$. So from the definition of $S_p(n)$ we have $S_p(p) = p^2 = dp$. This proves the case (1) of Lemma.

Now we prove the case (2) of Lemma. Using the same method of proving the case (1) of Lemma we can also deduce that if $p < d \leq 2p$, then $\left[\frac{d-1}{p^j}\right] = 1$, $\left[\frac{d-1}{p^j}\right] = 0, (j = 2,\ 3,\ \cdots\cdots)$, we have

$$\sum_{j=1}^{\infty} \left[\frac{(d-1)p}{p^j}\right] = d - 1 + \left[\frac{d-1}{p}\right] + \left[\frac{d-1}{p^2}\right] + \cdots = d.$$

That is means that $S_p(d) = (d-1)p$. From Theorem 1.7.2 of reference [6] we know that if $p < d \leq 2p$, then $p^d \| ((d-1)p)!$. That is, $S_p(d) = (d-1)p$. This proves the lemma.

Now we use this Lemma to complete the proof of the theorem.

First we prove the case (1) of Theorem. From the case (1) of Lemma we know that if $1 \leq n \leq p$, then

$$\sum_{d|n} S_p(d) = \sum_{d|n} dp = p \sum_{d|n} d = p\sigma(n).$$

Now we prove the case (2) of Theorem. We find that if $p < n \leq 2p$ and $d|n$, then there is only one divisor $d(> p)$ of n, i.e. $d = n$, the reason is that if $d > p$ and $d|n$, then $n = d,\ 2d,\ 3d,\ \cdots$, but $2d > 2p,\ \cdots$. So we may immediately deduce the following conclusion: if $p < n \leq 2p$, then

$$\sum_{d|n} S_p(d) = \sum_{\substack{d|n \\ 1 \leq d \leq p}} S_p(d) + \sum_{\substack{d|n \\ p < d \leq 2p}} S_p(d) = p \sum_{\substack{d|n \\ 1 \leq d \leq p}} d + S_p(n) = p\sigma(n) + (n-1)p.$$

This completes the proof of Theorem.

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Smarandache sums of products

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Introduction This paper deals with the sums of products of first n natural numbers, taken r at a time. Many interesting results about the summations are obtained. Mr. Ramasubramanian [1] has already made some work in this direction. This paper is an extension of his work.

In next part, the sums of odd and even natural numbers are discussed, and also of natural numbers, not necessarily beginning with one. After that, properties of sequences, arising out of these sums are obtained. Interestingly, the numbers thus obtained are Stirlings numbers.

1.1 Definition. The Smarandache sum of products is denoted by $S(n,r)$, and is defined as sum of products of first n natural numbers, taken r at a time, without repeatition, $r \leq n$.

For example:

$S(4,1) = 1+2+3+4 = 10$,

$S(4,2) = 1\cdot 2+1\cdot 3+1\cdot 4+2\cdot 3+2\cdot 4+3\cdot 4 = 35$,

$S(4,3) = 1\cdot 2\cdot 3+1\cdot 2\cdot 4+1\cdot 3\cdot 4+2\cdot 3\cdot 4 = 50$,

$S(4,4) = 1\cdot 2\cdot 3\cdot 4 = 24$.

We assume that $S(n,0) = 1$.

1.2 Following are some elementary properties of $S(n,r)$:

1. $S(n,n) = \llcorner n =$factorial $n = nS(n-1,n-1)$,

2. $S(n,1) = n(n+1)/2$; these are triangular numbers,

3. $(p+1)(p+2)(p+3)\ldots(p+n) = S(n,0)p^n + S(n,1)p^{n-1} + S(n,2)p^{n-2} + S(n,3)p^{n-3} + \cdots + S(n,n-1)p + S(n,n)$,

4. $S(n,0) + S(n,1) + S(n,2) + \cdots + S(n,n) = S(n+1,n+1) = \llcorner(n+1)$,

5. Number of terms in $S(n,r) = {}^n Cr$.

The 4th property can be obtained by putting $p = 1$ in the 3-rd property.

Verification of 4th property for $n = 5$.

$S(5,0) = 1$,

$S(5,1) = 1+2+3+4+5 = 15$,

$S(5,2) = 1\cdot 2+1\cdot 3+1\cdot 4+1\cdot 5+2\cdot 3+2\cdot 4+2\cdot 5+3\cdot 4+3\cdot 5+4\cdot 5 = 85$,

$S(5,3) = 1\cdot 2\cdot 3+1\cdot 2\cdot 4+1\cdot 2\cdot 5+1\cdot 3\cdot 4+1\cdot 3\cdot 5+2\cdot 3\cdot 4+2\cdot 3\cdot 5+3\cdot 4\cdot 5+2\cdot 4\cdot 5+2\cdot 4\cdot 5 = 225$,

$S(5,4) = 1\cdot 2\cdot 3\cdot 4+1\cdot 2\cdot 3\cdot 5+1\cdot 3\cdot 4\cdot 5+2\cdot 3\cdot 4\cdot 5+1\cdot 2\cdot 4\cdot 5 = 274$,

$S(5,5) = 1\cdot 2\cdot 3\cdot 4\cdot 5 = 120$.

Hence, left side is

$$1 + 15 + 85 + 225 + 274 + 120 = 720,$$

right side is

$$S(6, 6) = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720.$$

1.3 We have,

$$\begin{aligned}(p-1)(p-2)(p-3)\ldots(p-n) &= S(n,0) - S(n,1)p^n + S(n,2)p^{n-1} - S(n,3)p^{n-3} \\ &\quad + \ldots + S(n,n-1)p + (-1)^n S(n,n).\end{aligned}$$

Put $p = 1$,

$$S(n,0) - S(n,1) + S(n,2) - S(n,3) + \cdots + S(n,n-1) + (-1)^n S(n,n) = 0.$$

If n is even, then

$S(n,0) + S(n,2) + S(n,4) + \ldots + S(n,n) = S(n,1) + S(n,3) + S(n,5) + \ldots + S(n,n-1)$, for odd n,

$S(n,0) + S(n,2) + S(n,4) + \ldots + S(n,n-1) = S(n,1) + S(n,3) + S(n,5) + \ldots + S(n,n)$.

1.4 To verify $\mathbf{S(n,r) = S(n-1,r) + nS(n-1,r-1), r < n}$.

Put $r = 1$, then, the left side is

$$S(n,1) = n(n+1)/2.$$

Right side is

$$S(n-1,1) + nS(n-1,0) = (n-1)n/2 + n \cdot 1 = n \cdot (n+1)/2.$$

2.1 Reduction formula (I) for $\mathbf{S(n,r)}$.

Here we use the result of 1.4 repeatedly.

We have,

$$\begin{aligned}S(n,r) &= S(n-1,r) + nS(n-1,r-1), \\ S(n-1,r) &= S(n-2,r) + (n-1)S(n-2,r-1), \\ S(n-2,r) &= S(n-3,r) + (n-2)S(n-3,r-1), \\ S(n-3,r) &= S(n-4,r) + (n-1)S(n-4,r-1), \\ &\cdots \\ S(r+1,r) &= S(r,r) + (r+1)S(r,r-1).\end{aligned}$$

Adding, we get,

$$\begin{aligned}S(n,r) &= S(r,r) + (n)S(n-1,r-1) + (n-1)S(n-2,r-1) + (n-2)S(n-3,r-1) \\ &\quad + \ldots + (r+1)S(r,r-1)\end{aligned}$$

Verification:

Put $n = 5,\ r = 2$, the left side is

$$S(5,2) = 85,$$

right side is

$$S(2,2) + 5S(4,1) + 4S(3,1) + 3S(2,1) = 2 + 50 + 24 + 9 = 85,$$

2.2 Reduction formula (II) for S(n, r).

We have,

$$\begin{aligned} & (p+1)(p+2)(p+3)\ldots(p+n)(p+n+1) \\ = \ & S(n+1,0)p^{n+1} + S(n+1,1)p^n + S(n+1,2)p^{n-1} + S(n+1,3)p^{n-2} \\ & + \ldots + S(n+1,r+1)p^{n-r} + \ldots + S(n+1,n+1). \end{aligned} \tag{1}$$

Left side of (1) is

$$\begin{aligned} & (p+1)\{(p+1+1)(p+1+2)...(p+1+n-1)(p+1+n)\} \\ = \ & (p+1)\{S(n,0)(p+1)^n + S(n,1)(p+1)^{n-1} + \ldots + S(n,r)(p+1)^{n-r} \ldots + S(n,n)\} \\ = \ & S(n,0)(p+1)^{n+1} + S(n,1)(p+1)^n + \ldots + S(n,r)(p+1)^{n-r+1} + \ldots + S(n,n). \end{aligned}$$

Expanding each of

$(p+1)^{n+1}, (p+1)^n, (p+1)^{n-1}, \ldots, (p+1)^{n-r+1}$,

by binomial theorem, we get the left side of (1) is

$$\begin{aligned} & S(n,0)[C(n+1,0)p^{n+1} + C(n+1,1)p^n + \ldots + C(n+1,r+1)p^{n-r} \ldots + C(n+1,n+1)] \\ & + S(n,1)\{C(n,0)p^n + C(n,1)p^{n-1} + \ldots + C(n,r)p^{n-r} \ldots + C(n,n)\} \\ & + S(n,2)\{C(n-1,0)p^{n-1} + \ldots + C(n-1,r-1)p^{n-r} \ldots + C(n+1,n+1)\} + \ldots \\ & + S(n,r)\{C(n-r+1,1)p^{n-r} + \ldots\} \\ & + S(n,r+1)\{C(n-r,0)p^{n-r}\} \\ & + \ldots + S(n,n)(p+1), \end{aligned} \tag{2}$$

where $C(n,r)$ =combinations of n things, taken r at a time ($=^n C_r$).

Now, comparing the coefficients of p^{n-r} from right side of (1), and that from (2), we get,

$$\begin{aligned} S(n+1,r+1) \ = \ & C(n+1,r+1)S(n,0) + C(n,r)S(n,1) + C(n-1,r-1)S(n,2) \\ & + \ldots + C(n-r+1,1)S(n,r) + \ldots + S(n+1,r). \end{aligned}$$

because $C(n-r,0) = 1$.

Now,

$$S(n+1,r+1) = S(n,r+1) + (n+1)S(n,r),$$

from (1.4) above. Hence,

$$\begin{aligned} S(n,r+1) + (n+1)S(n,r) \ = \ & C(n+1,r+1)S(n,0) + C(n,r)S(n,1) + C(n-1,r-1) \\ & S(n,2) + \ldots + C(n-r+1,1)S(n,r) + \ldots + S(n+1,r), \end{aligned}$$

or

$$\begin{aligned}(n+1)S(n,r)-C(n-r+1,1)S(n,r) &= C(n+1,r+1)S(n,0)+C(n,r)S(n,1)+\ldots+\\ &\quad C(n-1,r-1)S(n,2)+\ldots+C(n-r+2,2)\\ &\quad S(n,r-1),\end{aligned}$$

or

$$\begin{aligned}S(n,r)[(n+1)-(n-r+1)] &= C(n+1,r+1)S(n,0)+C(n,r)S(n,1)+\ldots+\\ &\quad C(n-1,r-1)S(n,2)+\ldots+C(n-r+2,2)S(n,r-1),\end{aligned}$$

that is

$$\begin{aligned}r\cdot S(n,r) &= C(n+1,r+1)S(n,0)+C(n,r)S(n,1)+C(n-1,r-1)S(n,2)+\\ &\quad \ldots+C(n-r+2,2)S(n,r-1).\end{aligned}$$

This is the required result.

Verification:

Put $r=2$,

$$\begin{aligned}S(n,2) &= C(n+1,3)\cdot S(n,0)+C(n,2)\cdot S(n,1)\\ &= (n+1)n(n-1)/6+n(n-1)\cdot n\cdot(n+1)/2,\end{aligned}$$

or

$$S(n,2)=(n-1)\cdot n\cdot(n+1)\cdot(3n+2)/24.$$

For $n=5$,

$$S(5,2)=4\cdot 5\cdot 6\cdot 17/24=85,$$

which is true.

For $n=4$,

$$S(4,2)=3\cdot 4\cdot 5\cdot 14/24=35,$$

which is also true.

2.3 Reduction formula for $\mathbf{S(n,2)}$.

We have,

$$S(n,r)=S(n-1,r)+nS(n-1,r-1),\ r<n.$$

Put $r=2$,

$$S(n,2)=S(n-1,2)+nS(n-1,1),$$

$$S(n-1,2)=S(n-2,2)+(n-1)S(n-2,1),$$

$$S(n-2,2)=S(n-3,2)+(n-2)S(n-3,1),$$

$$\ldots$$

$$S(3,2)=S(2,2)+3S(2,1),$$

$$S(2,2) = 0 + 2S(1,1).$$

Now, $S(1,1) = 1$. Hence adding these results,

$$\begin{aligned} S(n,2) &= 2 + \sum_{3}^{n} pS(p-1,1)/2 \\ &= 2 + \sum_{3}^{n} p(p-1)p/2 \\ &= 2 + \sum_{1}^{n} (p^3 - p^2)/2 - (1/2)[(1^3 + 2^3) - (1^2 + 2^2)] \\ &= 2 + n^2(n+1)^2/8 - n(n+1)(2n+1)/12 - (9-5)/2 \\ &= (n-1)n(n+1)(3n+2)/24. \end{aligned}$$

This results is already obtain.

2.4 Similarly,

$$S(n,3) = C(n+1,4)3n(n+1)/64,$$

$$S(n,4) = C(n+1,5)(15n^3 + 15n^2 - 10n + 8)/48,$$

$$S(n,5) = C(n+1,6)(3n^4 + 2n^3 - 7n^2 - 6n)/16.$$

3.1 Definition.

$E(n,r)$ is sum of products of first n even natural numbers, taken r at a time.

$O(n,r)$ is sum of products of first n odd natural numbers, taken r at a time.

We have,

$$\begin{aligned} (p+2)\cdot(p+4)\cdot(p+6) &= p^3 + 12p^2 + 44p + 48 + \ldots \\ &= E(3,0)p^3 + E(3,1)p^2 + E(3,2)p + E(3,3), \end{aligned} \tag{3}$$

where,

$E(3,0) = 1$,

$E(3,1) = 2 + 4 + 6 = 12$,

$E(3,2) = 2\cdot 4 + 2\cdot 6 + 4\cdot 6 = 44$,

$E(3,3) = 2\cdot 4\cdot 6 = 48$.

Now, put $p = 1$ in (3),

Left side is

$$3\cdot 5\cdot 7 = O(3,3).$$

Hence,

$$O(3,3) = E(3,0) + E(3,1) + E(3,2) + E(3,3) = 105,$$

$$O(5,5) = E(4,0) + E(4,1) + E(4,2) + E(4,3) + E(4,4) = 945.$$

Similarly, we have,

$$E(3,3) = O(3,0) + O(3,1) + O(3,2) + O(3,3) = 48.$$

Therefore, in general, it can be conjectured that ,

$$O(n+1,n+1)=\sum_{r=0}^{n}E(n,r),$$

and

$$E(n,n)=\sum_{r=0}^{n}O(n,r).$$

3.2 Now we extend the definition of summation for a set of natural numbers, not necessarily beginning from 1.

$S[(p+1,p+n),r]$ is sum of products of n natural numbers, beginning from a natural number $p+1$, taken r at a time.

We state,

$$\begin{aligned} & S[(p+1,p+n),0]+S[(p+1,p+n),1]+S[(p+1,p+n),2] \\ & +\ldots+S[(p+1,p+n),r]+\ldots+S[(p+1,p+n),n] \\ = \; & S[(p+1,p+n+1),r]\ldots. \end{aligned} \tag{4}$$

Verification:

Put $p=n=3$.

To verify:

$$S[(3,6),0]+S[(3,6),1]+S[(3,6),2]+S[(3,6),3]+S[(3,6),4]=S[(4,7),4].$$

Left side is

$$\begin{aligned} & 1+(3+4+5+6)+(3\cdot4+3\cdot5+3\cdot6+4\cdot5+4\cdot6+5\cdot6) \\ & +(3\cdot4\cdot5+3\cdot4\cdot6+4\cdot5\cdot6+4\cdot5\cdot6)+(3\cdot4\cdot5\cdot6) \\ = \; & 1+18+119+342+360 \\ = \; & 840. \end{aligned}$$

Right side is

$$4\cdot5\cdot6\cdot7=840.$$

Hence verified. Hence the result (4) is true.

Similar results are true for odd and even integers,as

$$O[(2,7),0]+O[(2,7),1]+O[(2,7),2]+O[(2,7),3]+O[(2,7),4]=E[(3,8),4],$$

and

$$E[(2,7),0]+E[(2,7),1]+E[(2,7),2]+E[(2,7),3]+E[(2,7),4]=O[(3,8),4].$$

The verification of these results are simple.

3.3 To prove:

1. $S(n-1,1)=nS(n,1)-n$,
2. $S(n-1,2)=S(n,2)-nS(n,1)+n^2$,
3. $S(n-1,3)=S(n,3)-nS(n,2)+n^2S(n,1)-n^3$,

...

4. $S(n-1,r) = S(n,r) - nS(n,r-1) + n^2S(n,r-2) - n^3S(n,r-3) + \ldots$

Proof. We have

$$\begin{aligned} & (p+1)(p+2)(p+3)\ldots(p+n-1)(p+n) \\ = \quad & S(n,0)p^n + S(n,1)p^{n-1} + S(n,2)p^{n-2} + S(n,3)p^{n-3} + \ldots, \end{aligned} \tag{5}$$

divide both sides by $(p+n)$,

$$\begin{aligned} & (p+1)(p+2)(p+3)\ldots(p+n-1) \\ = \quad & [S(n,0)p^n + S(n,1)p^{n-1} + S(n,2)p^{n-2} + S(n,3)p^{n-3} + \ldots]/(p+n). \end{aligned} \tag{6}$$

Now using result (5), the left side of (6) is

$$S(n-1,0)p^{n-1} + S(n-1,1)p^{n-2} + S(n-1,2)p^{n-3} + S(n-1,3)p^{n-4} + \ldots. \tag{7}$$

By actual division, the right side of (6) is

$$p^{n-1} + p^{n-2} \cdot [S(n,1) - n] + p^{n-3}[S(n,2) - n \cdot S(n,1) + n] + \ldots. \tag{8}$$

Equating the coefficients of like powers of p from (7) and (8),
we have,

$S(n-1,1) = n \cdot S(n,1) - n,$
$S(n-1,2) = S(n,2) - n \cdot S(n,1) + n$
$S(n-1,3) = S(n,3) - n \cdot S(n,2) + n^2 \cdot S(n,1) - n^3 \cdots .$
$S(n-1,r) = S(n,r) - n \cdot S(n,r-1) + n^2S(n,r-2) - n^3 \cdot S(n,r-3) + \cdots .$
Verification :
For $n = 5$, and $r = 3$,
right side is

$$S(4,3) = 50,$$

also, left side is

$$S(5,3) - 5 \cdot S(5,2) + 52 \cdot S(5,1) - 53 = 225 - 5(85) + 25(15) = 50.$$

We have,

$$S(n,r) = S(n-1,r) + n \cdot S(n-1,r-1), r < n.$$

Also

$$S(n-1,r) = S(n-2,r) + (n-1) \cdot S(n-2,r-1).$$

Adding,

$$S(n,r) = S(n-2,r) + (n-1) \cdot S(n-2,r-1) + n \cdot S(n-1,r-1).$$

Again,

$$S(n-1,r-1) = S(n-2,r-1) + n - 1 \cdot S(n-2,r-2),$$

hence,

$$S(n,r) = S(n-2,r) + n - 1 \cdot S(n-2,r-1) + n \cdot S(n-2,r-1) + n \cdot (n-1)S(n-2,r-2),$$

$$S(n,r) = S(n-2,r) + (2n-1)\cdot S(n-2,r-1) + n\cdot(n-1)\cdot S(n-2,r-2).$$

Verification:

Put $n = 5$ and $r = 3$, the left side is

$$S(5,3) = 225,$$

right side is

$$S(3,3) + (9)\cdot S(3,2) + 5\cdot 4\cdot S(3,1) = 6 + 9.11 + 5.4.6 = 225,$$

hence verified.

4.1 Interestingly, the set of numbers $S(n,r)$, forms a sequence when r is fixed.

For,

$S(2,2) = 2, S(3,2) = 11, S(4,2) = 35, S(5,2) = 85, S(6,2) = 175\ldots$

The numbers

$$\{2, 11, 35, 85, 175, 322, 546, 870, 1320, 1925, 2717, 3731, \ldots,\}$$

are the **Stirling numbers** of first kind.

These numbers are the numbers of edges of a complete k-partite graph of order $S(k,1)$, that is of order $k(k+1)/2$.

The n-th term of this sequence is given by

$$\mathbf{a_n = a_{n-1} + [n(n+1)2]/2, n \geq 2},$$

and $a_1 = 2$.

4.2 A similar sequence for $S(n,3)$ is

$$\{6, 50, 225, 1960, 4536, 9450, 18150, 32670, \ldots\}$$

with similar properties , thus the sequences generated by $S(n,r)$ create additional good results. These are also **Stirling numbers**.

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On the solutions of an equation involving the Smarandache dual function

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Abstract In this paper, we use the elementary method to study the positive integer solutions of an equation involving the Smarandache dual function $\bar{s}_k(n)$, and give its all solutions.

Keywords Smarandache dual function, the positive integer solutions.

§1. Introduction

For any positive integer n, the famous Smarandache function $S(n)$ is defined by

$$S(n) = \min\{m \in N : n \mid m!\}.$$

For example, $S(1) = 1$, $S(2) = 2$, $S(3) = 3$, $S(4) = 4$, $S(5) = 5$, $S(6) = 3$, $S(7) = 7$, $S(8) = 4, \cdots$. This function was introduced by American-Romanian number theorist professor F.Smarandache, see reference [1]. About the arithmetical properties of $S(n)$, many scholars had studied it, and obtained some interesting conclusions, see references [2] and [3].

At the same time, many scholars also studied another function which have close relations with the Smarandache function. It is called the Smarandache ceil function $S_k(n)$, we define this arithmetic function as follows:

$$s_k(n) = \min\{m \in N : n \mid m^k\}.$$

For example, if $k = 3$, we have the sequence $\{s_3(n)\}$ $(n = 1, 2, 3, \cdots)$ as following: $s_3(1) = 1$, $s_3(2) = 2$, $s_3(3) = 3$, $s_3(4) = 2$, $s_3(5) = 5$, $s_3(6) = 6$, $s_3(7) = 7$, $s_3(8) = 2$, $\cdots$. This arithmetical function is a multiplicative function, and has many interesting properties, so it had be studied by many people, see references [4] and [5].

Similarly, we will introduce the Smarandache dual function $\bar{s}_k(n)$ which denotes the greatest positive integer m such that $m^k|n$, where n denotes any positive integer. That is,

$$\bar{s}_k(n) = \max\{m \in N : m^k|n\}.$$

It is easy to calculating that $\bar{s}_3(1) = 1$, $\bar{s}_3(2) = 1$, $\bar{s}_3(3) = 1$, $\bar{s}_3(4) = 1$, $\bar{s}_3(5) = 1$, $\bar{s}_3(6) = 1, \bar{s}_3(7) = 1$, $\bar{s}_3(8) = 2, \cdots$. About this function, Lu Yaming [6] studied the asymptotic properties of the summation $\sum_{n \le x} d(\bar{s}_k(n))$ by using the elementary methods, and obtained an

interesting asymptotic formula:

$$\sum_{n\leq x} d\left(\bar{s}_k(n)\right) = \zeta(k)x + \zeta\left(\frac{1}{k}\right)x^{\frac{1}{k}} + O\left(x^{\frac{1}{k+1}}\right).$$

Ding Liping [7] also studied the mean value properties of the summation $\sum_{n\leq x}\bar{s}_2(a(n))$, and give a sharper mean value theorem:

$$\sum_{n\leq x}\bar{s}_2(a(n)) = \frac{x^2\pi^4}{315}\prod_p\left(1+\frac{1}{p^4+p^3}\right) + O\left(x^{\frac{3}{2}+\varepsilon}\right),$$

where $a(n)$ denotes the cubic complements of n, $\zeta(s)$ is the Riemann zeta-function.

On the other hand, we let $\Omega(n)$ denotes the number of the prime divisors of n, including multiple numbers. If $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$ denotes the factorization of n into prime powers, then

$$\Omega(n) = \alpha_1 + \alpha_2\cdots + \alpha_r,$$

we have $\Omega(1) = 0$, $\Omega(2) = 1$, $\Omega(3) = 1$, $\Omega(4) = 2$, $\Omega(5) = 1$, $\Omega(6) = 2$, $\Omega(7) = 1$, $\Omega(8) = 3$, $\cdots$. In [8], the author studied the positive integer solutions of the equation

$$\bar{s}_3(1) + \bar{s}_3(2) + \cdots + \bar{s}_3(n) = 3\Omega(n),$$

and proved that this equation has three solutions $n = 3$, 6 and 8.

This paper, as a note of [8], we shall prove a general conclusion for the equation

$$\bar{s}_k(1) + \bar{s}_k(2) + \cdots + \bar{s}_k(n) = k\Omega(n).$$

That is, we shall prove the following:

Theorem. For all positive integer n, the equation

$$\bar{s}_k(1) + \bar{s}_k(2) + \cdots + \bar{s}_k(n) = k\Omega(n)$$

has no solution if $k = 1$. If $k \geq 2$, then the equation has at least one positive integer solution. They are

i) $n = 2$, if $k = 2$;

ii) $n = 3$, 6, 8, if $k = 3$;

iii) If $k \geq 4$ and $n < 2^k$, then the equation has solution if and only if $n = k\Omega(n)$. Especially, if $k = p$ be any prime, then $n = p$ and $2p$ are two solutions of the equation.

§2. Some lemmas

To complete the proof of the theorem, we need the following lemmas:

Lemma 1. For all positive integer n, the equation

$$\bar{s}_1(1) + \bar{s}_1(2) + \cdots + \bar{s}_1(n) = \Omega(n),$$

has no positive integer solution.

Proof. For any positive integer n, if $k = 1$, then we have

$$s_1(n) = n,$$

and then $\sum_{i=1}^{n} s_1(i) \geq n$.

If $n = 1$, then $\bar{s}_1(1) = 1$ and $\Omega(1) = 0$, this time we know that the equation has no positive integer solution; If $n \geq 2$ $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ and p_i is a prime, then

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} > \alpha_1 \cdot \alpha_2 \cdot \cdots \alpha_r.$$

Note that if $a > 1$, $b > 1$, then $a \cdot b \geq a + b$, so we have

$$\sum_{i=1}^{n} s_1(i) > \alpha_1 \cdot \alpha_2 \cdot \cdots \alpha_r > \alpha_1 + \alpha_2 + \cdots + \alpha_r = \Omega(n).$$

So for all positive integer n, from the above formula we know that the equation $\bar{s}_1(1) + \bar{s}_1(2) + \cdots + \bar{s}_1(n) = \Omega(n)$ has no positive integer solution.

This proved Lemma 1.

Lemma 2. For all positive integer n, the equation

$$\bar{s}_2(1) + \bar{s}_2(2) + \cdots + \bar{s}_2(n) = 2\Omega(n)$$

has one solution $n = 2$;

The equation

$$\bar{s}_3(1) + \bar{s}_3(2) + \cdots + \bar{s}_3(n) = 3\Omega(n)$$

has three solutions, $n = 3$, 6, 8.

Proof. The second result can be deduced directly from reference [8]. By the same way we can also get the first result of Lemma 2.

Lemma 3. For all integers $k \geq 4$, we have $2^{k+1} > k(k+1)$.

Proof. Now we prove this inequality by mathematical induction on k.

For $k = 4$, it holds trivially.

Suppose, then, that the formula holds for all integers $< k$. That is, for all integers $m < k$, we have $2^{m+1} > m(m+1)$.

Note that $2m(m+1) > (m+1)(m+2)$, from the inductive hypothesis we have

$$2^{m+1+1} > 2m(m+1) > (m+1)(m+2).$$

That is means, the inequality also holds for $m + 1$.

This proved Lemma 3.

Lemma 4. If $k \geq 4$, then for all positive integer $n \geq 2^k$, we have

$$\bar{s}_k(1) + \bar{s}_k(2) + \cdots + \bar{s}_k(n) > k\Omega(n).$$

Proof. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the factorization of n into prime powers, then we have

$$\bar{s}_k(1) + \bar{s}_k(2) + \cdots + \bar{s}_k(n) > n \quad \text{if} \quad n \geq 2^k.$$

From the definition of $\Omega(n)$, we have

$$\Omega(n) = \alpha_1 + \alpha_2 \cdots + \alpha_r.$$

So to complete the proof of the lemma, we only to prove the following inequality:

$$p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} > k(\alpha_1 + \alpha_2 \cdots + \alpha_r). \tag{1}$$

Now we prove (1) by mathematical induction on r.

i) If $r = 1$, then $n = p_1^{\alpha_1}$.

a. If $p_1 = 2$, since $2^{k+1} > 2^k$, then we have $\alpha_1 \geq k+1$, and Lemma 3 tell us $2^{k+1} > k(k+1)$. Hence

$$2^{\alpha_1} > k\alpha_1.$$

b. If $p_1 \geq 3$, then $p_1^k > 2^k$. So we have $\alpha_1 \geq k$. If $k = 4$, we have $p^4 > 4^2$, and if $k > 4$, we have $p^k > 4^k > k^2$. So we know

$$p_1^k > k \cdot k.$$

Hence

$$p_1^{\alpha_1} > k\alpha_1.$$

This proved that the lemma 4 holds for $r = 1$.

ii) Now we assume that the (1) holds for r (≥ 2), we shall prove that it is also holds for $r + 1$.

From the inductive hypothesis, we have

$$p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p_{r+1}^{\alpha_{r+1}} > k(\alpha_1 + \alpha_2 + \cdots + \alpha_r) \cdot p_{r+1}^{\alpha_{r+1}}.$$

Since p_{r+1} is a prime, then

$$p_{r+1}^{\alpha_{r+1}} > \alpha_{r+1} + 1.$$

From above we obtain

$$p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p_{r+1}^{\alpha_{r+1}} > k(\alpha_1 + \alpha_2 + \cdots + \alpha_r) \cdot (\alpha_{r+1} + 1).$$

Note that if $a > 1$, $b > 1$, then $a \cdot b \geq a + b$, so we have

$$(\alpha_1 + \alpha_2 \cdots + \alpha_r) \cdot (\alpha_{r+1} + 1) \geq \alpha_1 + \alpha_2 + \cdots + \alpha_r + \alpha_{r+1} + 1 > \alpha_1 + \alpha_2 + \cdots + \alpha_r + \alpha_{r+1}.$$

So

$$p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p_{r+1}^{\alpha_{r+1}} > k(\alpha_1 + \alpha_2 + \cdots + \alpha_r + \alpha_{r+1}).$$

This completes the proof of Lemma 4.

§3. Proof of the theorem

In this section, we shall complete the proof of the theorem. According to the definition of $\bar{s}_k(n)$ and the results of Lemma 1 and Lemma 2, we only to prove the following case: when $k \geq 4$, whether there exists finite solutions for the equation

$$\bar{s}_k(1) + \bar{s}_k(2) + \cdots + \bar{s}_k(n) = k\Omega(n).$$

First we separate all positive integer into two cases:
1. If $n < 2^k$, then from the definition of $\bar{s}_k(n)$ and $\Omega(n)$, we have $\bar{s}_k(n) = 1$, so the equation $\bar{s}_k(1) + \bar{s}_k(2) + \cdots + \bar{s}_k(n) = k\Omega(n)$ become the form $n = k\Omega(n)$. Hence if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, then $n = k(\alpha_1 + \alpha_2 + \cdots + \alpha_r)$ are the positive integer solutions of the equation.
2. If $n \geq 2^k$, then from Lemma 4 we know that the equation has no positive integer solutions. Combining all the above cases we have the following conclusion:

The equation

$$\bar{s}_k(1) + \bar{s}_k(2) + \cdots + \bar{s}_k(n) = k\Omega(n)$$

has no solution if $k = 1$. If $k \geq 2$, then the equation has positive integer solutions. They are
i) $n = 2$, if $k = 2$;
ii) $n = 3$, 6, 8, if $k = 3$;
iii) If $k \geq 4$ and $n < 2^k$, then the equation has solution if and only if $n = k\Omega(n)$.

Note: It is clear that if $k = p$ be a prime, then $n = p$ or $2p$ are two solutions of the equation.

This completes the proof of Theorem.

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An infinite series involving the Smarandache power function $SP(n)$

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Abstract For any positive integer n, the Smarandache power function $SP(n)$ is defined as the smallest positive integer m such that $n|m^m$, where m and n have the same prime divisors. The main purpose of this paper is using the elementary methods to study the convergent properties of an infinite series involving the Smarandache power function $SP(n)$, and give some interesting identities.

Keywords Smarandache power function, infinite series, the Riemann zeta-function.

§1. Introduction and Results

For any positive integer n, we define the Smarandache power function $SP(n)$ as the smallest positive integer m such that $n|m^m$, where n and m have the same prime divisors. That is,

$$SP(n) = \min\left\{m : n|m^m, m \in \mathrm{N}, \prod_{p|n} p = \prod_{p|m} p\right\}.$$

If n runs through all natural numbers, then we can get the Smarandache power function sequence $\{SP(n)\}$: 1, 2, 3, 2, 5, 6, 7, 4, 3, 10, 11, 6, 13, 14, 15, 4, 17, 6, 19, 10, $\cdots$.

In reference [1], Professor F.Smarandache asked us to study the properties of the sequence $\{SP(n)\}$. From the definition of $SP(n)$ we can easily get the following conclusions:
If $n = p^\alpha$, where p be a prime, then we have

$$SP(n) = \begin{cases} p, & \text{if } 1 \le \alpha \le p; \\ p^2, & \text{if } p+1 \le \alpha \le 2p^2; \\ p^3, & \text{if } 2p^2+1 \le \alpha \le 3p^3; \\ \cdots & \\ p^\alpha, & \text{if } (\alpha-1)p^\alpha + 1 \le \alpha \le \alpha p^\alpha. \end{cases}$$

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ denotes the factorization of n into prime powers. If $\alpha_i \le p_i$ for all α_i $(i = 1, 2, \cdots, r)$, then we have $SP(n) = U(n)$, where $U(n) = \prod_{p|n} p$, $\prod_{p|n}$ denotes the product over all different prime divisors of n. It is clear that $SP(n)$ is not a multiplicative function. For example, $SP(8) = 4$, $SP(3) = 3$, $SP(24) = 6 \neq SP(3) \times SP(8)$. But for almost m and n with

$(m,n)=1$, we have $SP(mn)=SP(m)\cdot SP(n)$. In reference [2], Dr. Zhefeng Xu had studied the mean value properties of $SP(n)$, and obtained some sharper asymptotic formulas, one of them as follows:

$$\sum_{n\le x} SP(n)=\frac{1}{2}x^2\prod_p\left(1-\frac{1}{p(p+1)}\right)+O\left(x^{\frac{3}{2}+\epsilon}\right),$$

where ϵ denotes any fixed positive number, and $\prod_p$ denotes the product over all primes.

In this paper, we shall use the elementary methods to study the convergent properties of an infinite series involving the Smarandache power function $SP(n)$, and give some interesting identities. That is, we shall prove the following:

Theorem. For any complex number s with $\mathrm{Re}(s)>1$, we have the identities

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\mu(n)}{(SP(n^k))^s}=\begin{cases}\frac{2^s+1}{2^s-1}\frac{1}{\zeta(s)}, & \text{if } k=1,2;\\ \frac{2^s+1}{2^s-1}\frac{1}{\zeta(s)}-\frac{2^s-1}{4^s}, & \text{if } k=3;\\ \frac{2^s+1}{2^s-1}\frac{1}{\zeta(s)}-\frac{2^s-1}{4^s}+\frac{3^s-1}{9^s}, & \text{if } k=4,5.\end{cases}$$

where $\mu(n)$ denotes the Möbius function, and $\zeta(s)$ denotes the Riemann zeta-function.

Note that $\zeta(2)=\frac{\pi^2}{6}$, $\zeta(4)=\frac{\pi^4}{90}$, taking $s=2$ and 4 in our Theorem we may immediately deduce the following identities:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\mu(n)}{(SP(n^k))^2}=\begin{cases}\frac{10}{\pi^2}, & \text{if } k=1,2;\\ \frac{10}{\pi^2}-\frac{3}{16}, & \text{if } k=3;\\ \frac{10}{\pi^2}-\frac{3}{16}+\frac{8}{81}, & \text{if } k=4,5.\end{cases}$$

and

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\mu(n)}{(SP(n^k))^4}=\begin{cases}\frac{102}{\pi^4}, & \text{if } k=1,2;\\ \frac{102}{\pi^4}-\frac{15}{256}, & \text{if } k=3;\\ \frac{102}{\pi^4}-\frac{15}{256}+\frac{80}{6561}, & \text{if } k=4,5.\end{cases}$$

Note: For general integer $k\ge 6$, using our method we can also give a calculating formula for $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\mu(n)}{(SP(n^k))^s}$, but in this case, the conclusion is more complicate.

§2. Proof of the theorem

In this section, we shall prove our Theorem directly. Note that $\mu(2m)=0$ if m be an even number. If m be a odd number, then $\mu(2m)=-\mu(m)$. So we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\mu(n)}{(SP(n^k))^s} &= \sum_{m=1}^{\infty}\frac{\mu(2m-1)}{(SP((2m-1)^k))^s}-\sum_{m=1}^{\infty}\frac{\mu(2m)}{(SP((2m)^k))^s}\\ &= \sum_{m=1}^{\infty}\frac{\mu(2m-1)}{(SP((2m-1)^k))^s}+\sum_{m=1}^{\infty}\frac{\mu(2m-1)}{(SP(2^k(2m-1)^k))^s}. \qquad (1)\end{aligned}$$

If $k = 1$ or 2, then $\dfrac{\mu(n)}{SP(n^k)}$ is a multiplicative function. In fact for any positive integers m and n with $(m, n) = 1$, if $\mu(mn) = 0$, then $\mu(m) = 0$ or $\mu(n) = 0$, i.e. $\mu(m)\mu(n) = 0$. So

$$\frac{\mu(mn)}{SP(m^k n^k)} = \frac{\mu(m)}{SP(m^k)}\frac{\mu(n)}{SP(n^k)} = \frac{\mu(m)}{m}\frac{\mu(n)}{n}.$$

If $\mu(mn) \neq 0$, then $\mu(m) \neq 0$, $\mu(n) \neq 0$, and more $SP(m^k n^k) = SP(m^k)SP(n^k) = mn$. Thus, $\dfrac{\mu(n)}{SP(n^k)} = \dfrac{\mu(n)}{n}$ is a multiplicative function, if $k = 1$ or 2.
Now if $k = 1$ or 2, note that the identity (see Theorem 11.7 of [3])

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right) = \frac{1}{\zeta(s)},$$

where $\zeta(s)$ is the Riemann zeta-function, then from (1), the Euler product formula (see Theorem 11.6 of [3]) and the multiplicative properties of $\dfrac{\mu(n)}{(SP(n^k))^s}$ we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\mu(n)}{(SP(n^k))^s} &= \sum_{m=1}^{\infty} \frac{\mu(2m-1)}{(SP((2m-1)^k))^s} + \sum_{m=1}^{\infty} \frac{\mu(2m-1)}{(SP(2^k(2m-1)^k))^s} \\
&= \prod_{p\neq 2} \left(1 - \frac{1}{p^s}\right) + \frac{1}{2^s}\prod_{p\neq 2} \left(1 - \frac{1}{p^s}\right) \\
&= \frac{2^s+1}{2^s}\frac{2^s}{2^s-1}\prod_p \left(1 - \frac{1}{p^s}\right) = \frac{2^s+1}{2^s-1}\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \\
&= \frac{2^s+1}{2^s-1}\frac{1}{\zeta(s)}.
\end{aligned}$$

This proves the first formula of our Theorem.

If $k = 3$, note that $SP(2^3) = 4$, so $\dfrac{\mu(1)}{(SP(2^3))^s} = \dfrac{1}{4^s}$, and $\dfrac{\mu(2m-1)}{(SP(2^3(2m-1)^3))^s} = \dfrac{\mu(2m-1)}{2^s(2m-1)^s}$ for all $m > 1$. Then from (1) and the method of proving the first part of our Theorem we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\mu(n)}{(SP(n^3))^s} &= \sum_{m=1}^{\infty} \frac{\mu(2m-1)}{(SP((2m-1)^3))^s} + \sum_{m=1}^{\infty} \frac{\mu(2m-1)}{(SP(2^3(2m-1)^3))^s} \\
&= \sum_{m=1}^{\infty} \frac{\mu(2m-1)}{(SP((2m-1)^3))^s} + \frac{1}{4^s} - \frac{1}{2^s} + \sum_{m=1}^{\infty} \frac{\mu(2m-1)}{2^s(2m-1)^s} \\
&= \prod_{p\neq 2} \left(1 - \frac{1}{p^s}\right) - \frac{2^s-1}{4^s} + \frac{1}{2^s}\prod_{p\neq 2} \left(1 - \frac{1}{p^s}\right) \\
&= \frac{2^s+1}{2^s}\frac{2^s}{2^s-1}\prod_p \left(1 - \frac{1}{p^s}\right) - \frac{2^s-1}{4^s} \\
&= \frac{2^s+1}{2^s-1}\frac{1}{\zeta(s)} - \frac{2^s-1}{4^s}.
\end{aligned}$$

This proves the second formula of Theorem.

Now we prove the third formula of our Theorem.

For $k = 4$ or 5, note that $SP(2^k) = 4$, $SP(3^k) = 9$, i.e., $\frac{\mu(1)}{(SP(2^k))^s} = \frac{1}{4^s}$, $\frac{\mu(3)}{(SP(3^k))^s} = -\frac{1}{9^s}$, and $\frac{\mu(2m-1)}{(SP(2^k(2m-1)^k))^s} = \frac{\mu(2m-1)}{2^s(2m-1)^s}$ for all $m > 2$, $\frac{\mu(2m-1)}{(SP((2m-1)^k))^s} = \frac{\mu(2m-1)}{(2m-1)^s}$ for all $m \geq 2$. So from the Euler product formula and (1) we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\mu(n)}{(SP(n^k))^s} &= \sum_{m=1}^{\infty} \frac{\mu(2m-1)}{(SP((2m-1)^k))^s} + \sum_{m=1}^{\infty} \frac{\mu(2m-1)}{(SP(2^k(2m-1)^k))^s} \\
&= -\frac{1}{9^s} + \frac{1}{3^s} + \sum_{m=1}^{\infty} \frac{\mu(2m-1)}{(2m-1)^s} + \frac{1}{4^s} - \frac{1}{2^s} + \sum_{m=1}^{\infty} \frac{\mu(2m-1)}{2^s(2m-1)^s} \\
&= -\frac{1}{9^s} + \frac{1}{3^s} + \prod_{p\neq 2}\left(1-\frac{1}{p^s}\right) - \frac{2^s-1}{4^s} + \frac{1}{2^s}\prod_{p\neq 2}\left(1-\frac{1}{p^s}\right) \\
&= \frac{2^s+1}{2^s}\frac{2^s}{2^s-1}\prod_{p}\left(1-\frac{1}{p^s}\right) - \frac{2^s-1}{4^s} + \frac{3^s-1}{9^s} \\
&= \frac{2^s+1}{2^s-1}\frac{1}{\zeta(s)} - \frac{2^s-1}{4^s} + \frac{3^s-1}{9^s}.
\end{aligned}$$

This completes the proof of our Theorem.

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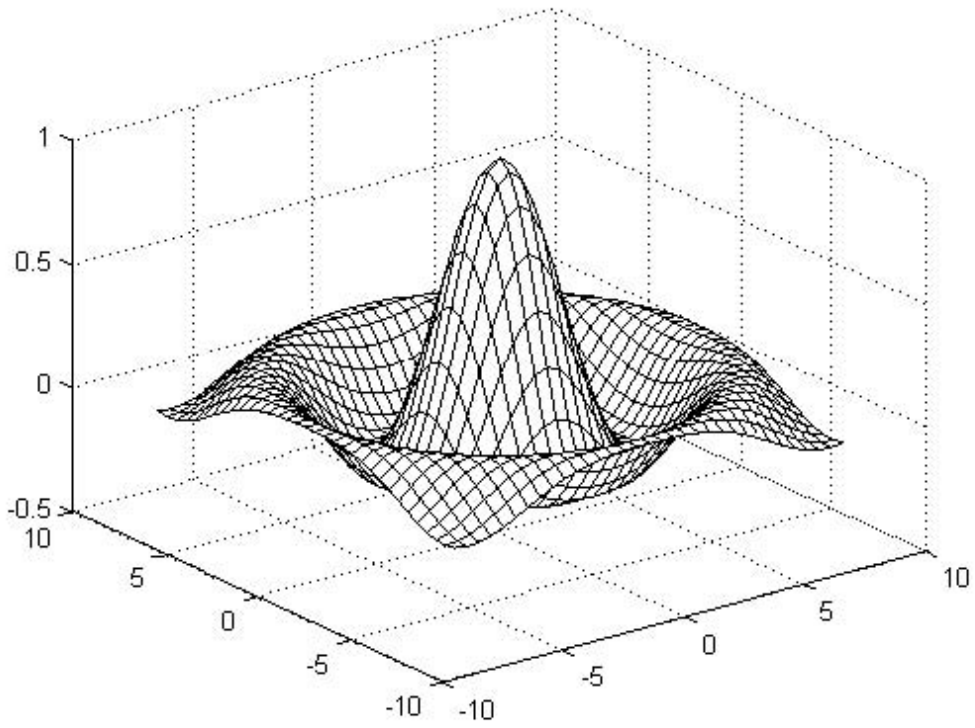

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